External and Internal Flows

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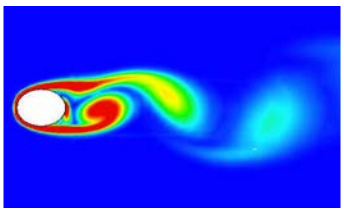
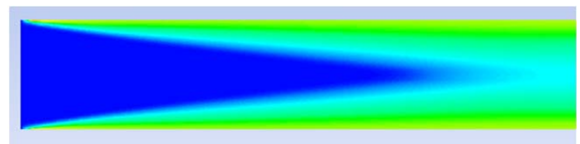
External Flows:
- Flow *around* a body
- Boundary layer develops freely without constraints from the geometry

Eg. Cylinder flow



Internal Flows:
- Flow *inside* a body
- Boundary layer is unable to develop without eventually being constrained

Eg. Pipe flow



Laminar Flow over a Flat Plate

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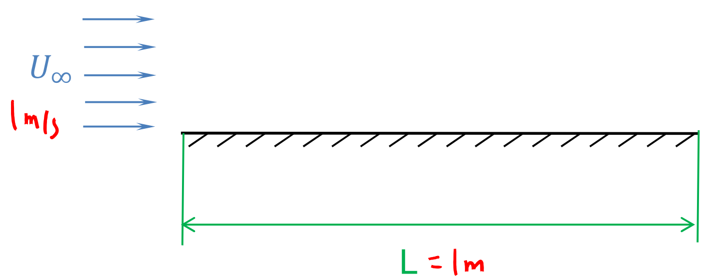
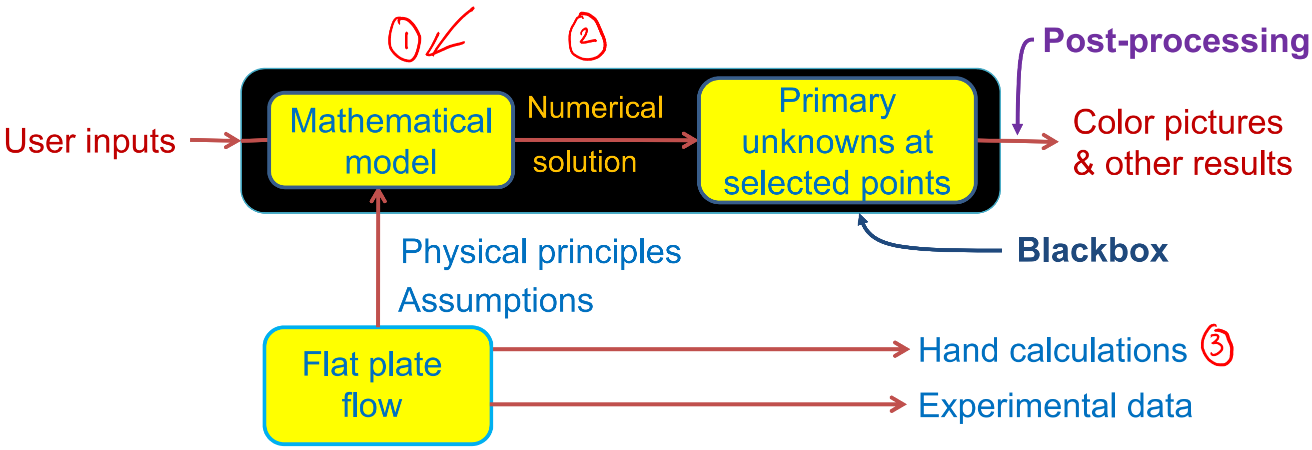


$$Re_L = \frac{\overset{1\,\frac{kg}{m^3}}{\rho} U_\infty L}{\underset{10^{-4}\,\frac{kg}{m\text{-}s}}{\mu}} = 10{,}000$$

- Simplest external flow
- We'll solve this flow numerically using ANSYS FLUENT
  - Solving Navier-Stokes equations, not boundary layer equations
- Compare FLUENT results with boundary layer theory
  - Verify FLUENT solution
  - Better understand boundary layer theory

What's Inside the CFD Blackbox?



Mathematical Model

- Boundary value problem
  1. Governing equations PDEs
     - Defined in a domain
  2. Boundary conditions
     - Defined at the edges of the domain

Governing Equations

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- Continuity
  - $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$
- $\vec{F} = m\,\vec{a}$ applied to a vanishingly small chunk of fluid
  - $\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$
  - $\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$

Unknowns:

$u(x,y)$, $v(x,y)$, $p(x,y)$

v_x v_y

Assumptions:

2D, steady, incompressible, laminar, Newtonian

Boundary Conditions

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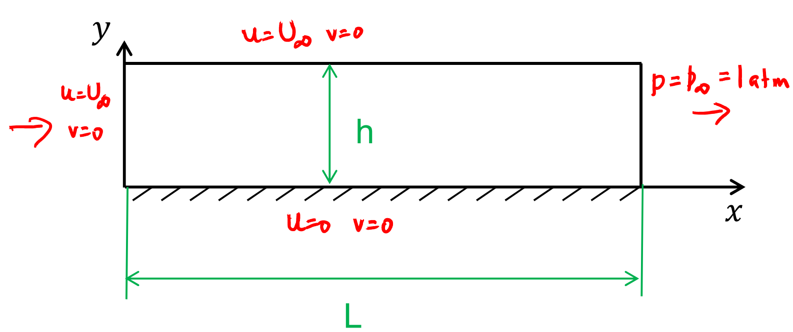


h is picked by user
- Need to verify that choice of *h* doesn't affect the solution

Need to also check the effect of moving the left boundary

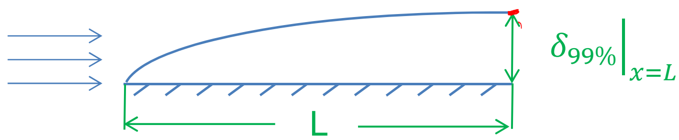
Investigating effect of outer boundaries is a basic check for all external flows

Domain: Picking h

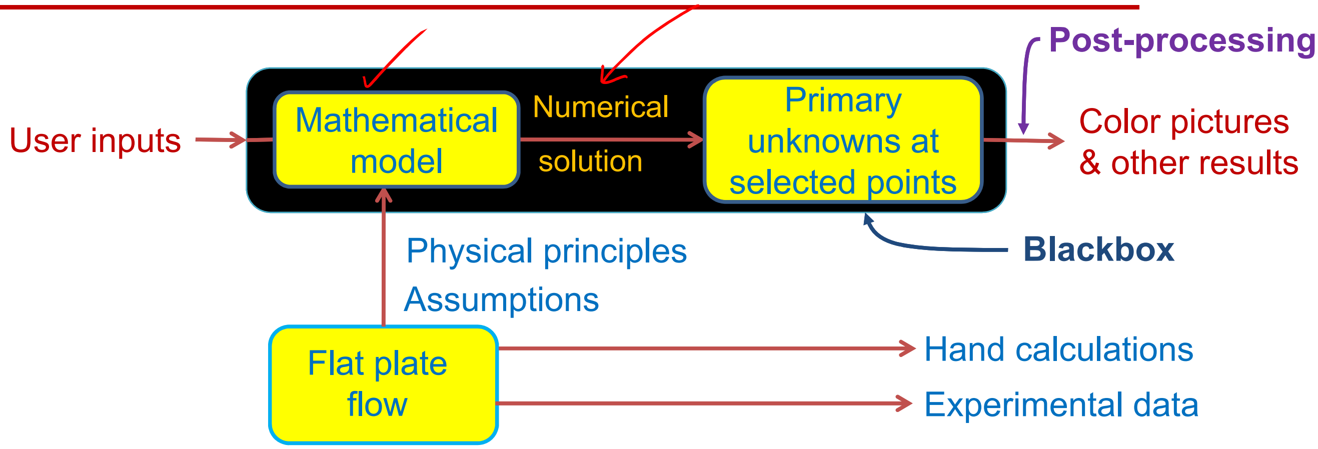
Pick $h \sim 10\ \delta_{99\%}|_{x=L}$

$h \sim 10 * 0.05 = 0.5m$

$$\frac{\delta_{99\%}}{L} = \frac{5}{\sqrt{Re_L}} = \frac{5}{\sqrt{10,000}} = 0.05$$



What's Inside the CFD Blackbox?



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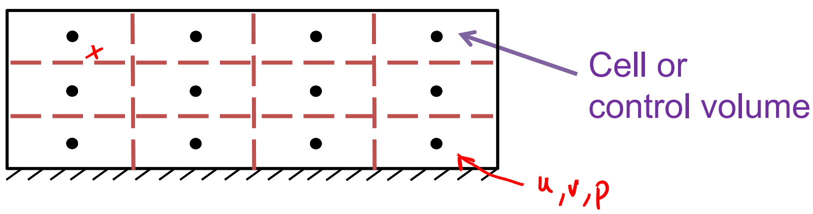
Finite Volume Method

- Divide the domain into multiple control volumes or “cells”
- Reduce the problem to determining velocity and pressure values at cell centers $u(x,y), v(x,y), p(x,y) \rightarrow 12 * 3 = 36$ values
- Use interpolation of cell center values to determine values at other locations within a cell



“Discretization”

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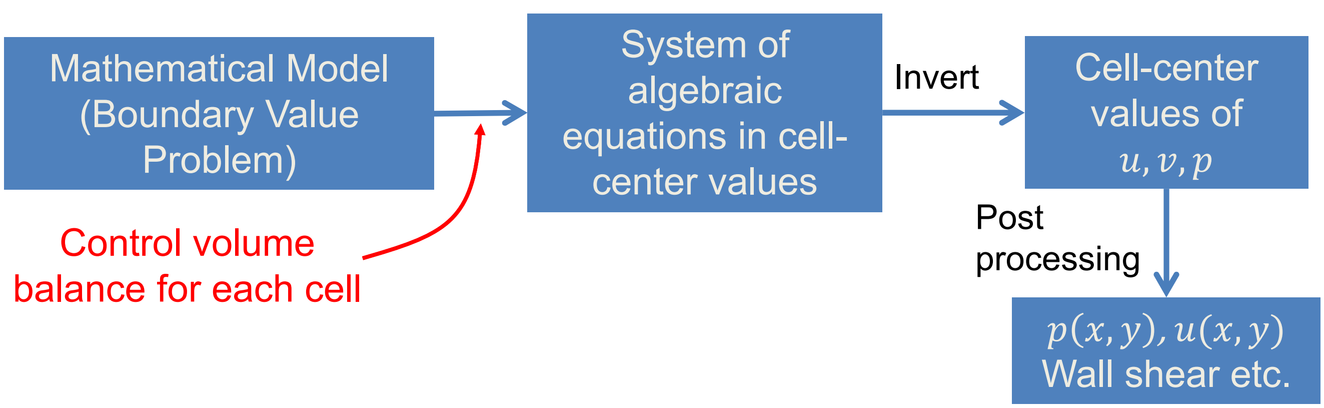
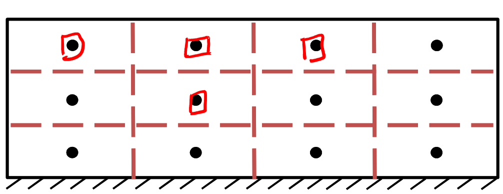
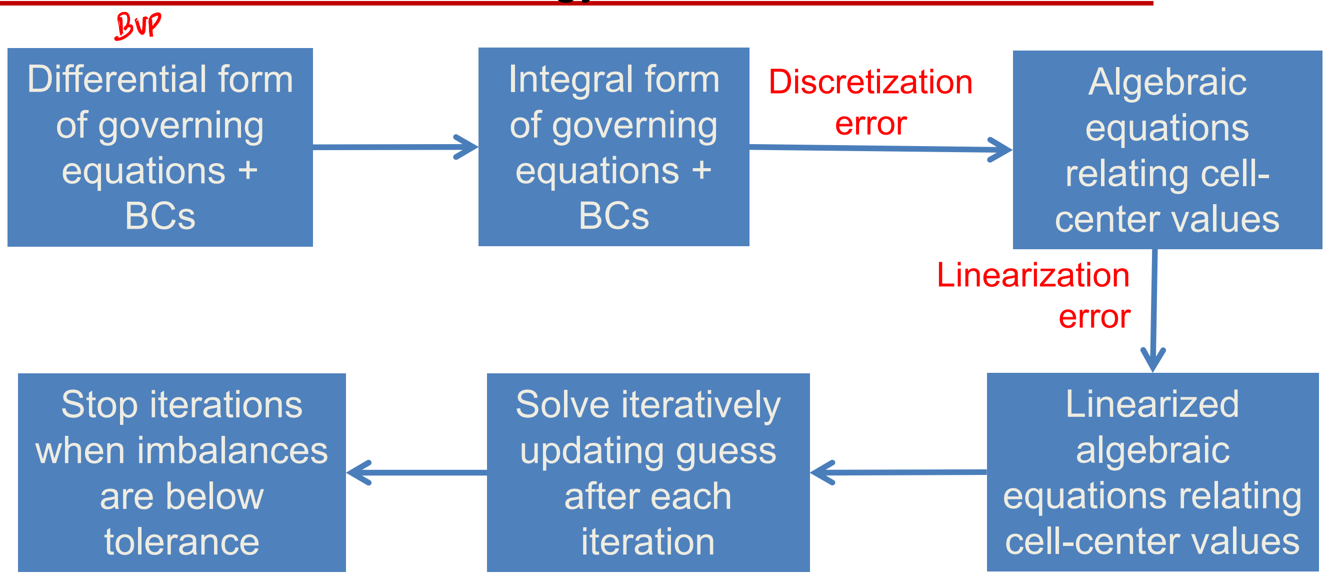
How to Find Velocity and Pressure at Cell Centers?



Each algebraic equation will relate a cell-center value to its neighbors

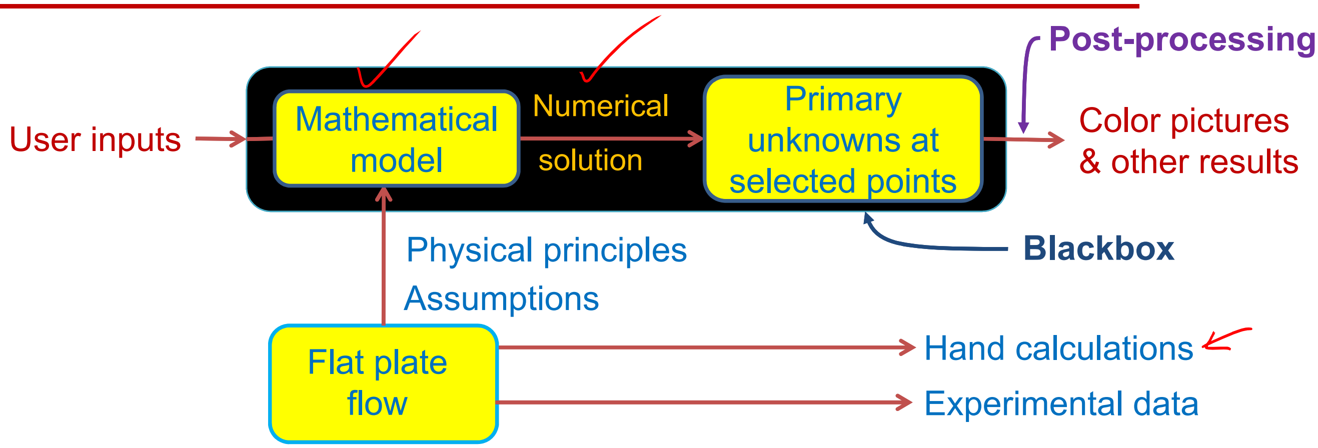


Numerical Solution Strategy: Overview



For more details, see online lectures at https://goo.gl/cfiKHq

What's Inside the CFD Blackbox?



Boundary Layer Theory

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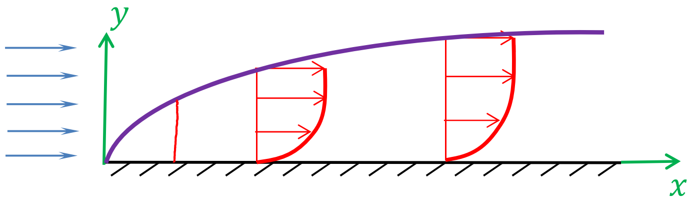


Similarity principle can be used to reduce 2 PDE's to 1 ODE

$$F''' + F\,F'' = 1$$

$$F(0) = F'(0) = 0 \qquad F'(\infty) = 1$$

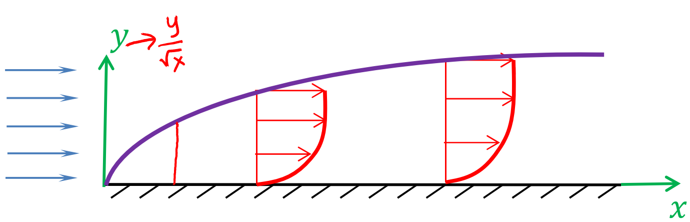
$$\Rightarrow C_D = \frac{Drag}{0.5\,\rho U_\infty^2 L} = \frac{1.328}{\sqrt{Re_L}}$$

$$u \gg v \qquad \frac{\partial u}{\partial x} \ll \frac{\partial u}{\partial y} \qquad \frac{\partial v}{\partial x} \ll \frac{\partial v}{\partial y}$$

- $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$
- $\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\cancel{\frac{\partial p}{\partial x}} + \mu\left(\cancel{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2}\right)$
- $\rho\left(\cancel{u\frac{\partial v}{\partial x}} + \cancel{v\frac{\partial v}{\partial y}}\right) = -\frac{\partial p}{\partial y} + \mu\left(\cancel{\frac{\partial^2 v}{\partial x^2}} + \cancel{\frac{\partial^2 v}{\partial y^2}}\right)$

Boundary Layer Theory

Cornell Engineering



Similarity principle can be used to reduce 2 PDE's to 1 ODE

$$2F''' + F\,F'' = 1$$

$$F(0) = F'(0) = 0 \qquad F'(\infty) = 1$$

$$\Rightarrow C_D = \frac{Drag}{0.5\,\rho U_\infty^2 L} = \frac{1.328}{\sqrt{Re_L}}$$

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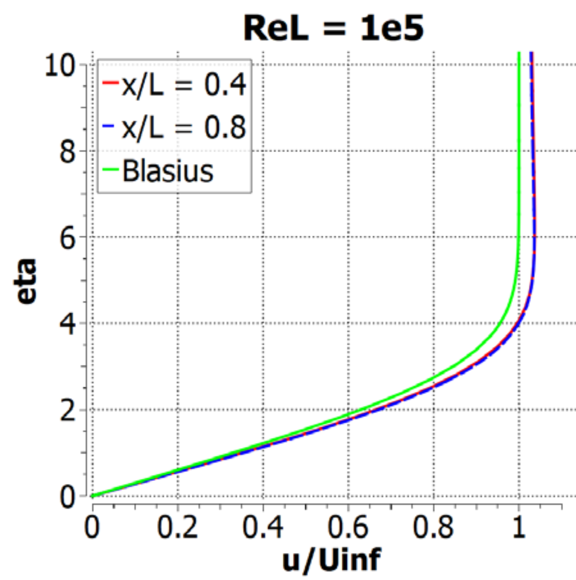
$$F = F(\eta) \qquad \eta = y\sqrt{\frac{\rho U_\infty}{\mu\, x}}$$

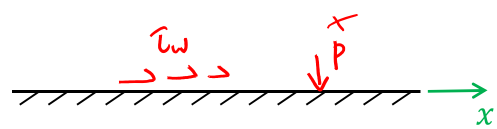
$$F'(\eta) = \frac{u}{u_\infty}$$

Comparison of FLUENT and Blasius Solutions



Drag Coefficient

- FLUENT integrates the wall shear and pressure to find the drag force
  - $D = \int_0^L \tau_w \; b \; dx$ for flat plate
  - *b* is depth normal to screen and taken as 1



- Drag coefficient
  - $C_D = \dfrac{D}{0.5\,\rho U_\infty^2\, L \cancel{b}}$
  - In FLUENT, $C_D = \dfrac{D}{0.5\,\rho_{ref} V_{ref}^2\, A_{ref}}$

Reference Values

Area (m2)	1	A_{ref}
Density (kg/m3)	1	ρ_{ref}
Depth (m)	1	
Enthalpy (j/kg)	0	
Length (m)	1	
Pressure (pascal)	0	
Temperature (k)	288.16	
Velocity (m/s)	1	V_{ref}
Viscosity (kg/m-s)	1.7894e-05	
Ratio of Specific Heats	1.4	

Laminar Flow over a Flat Plate with Heating

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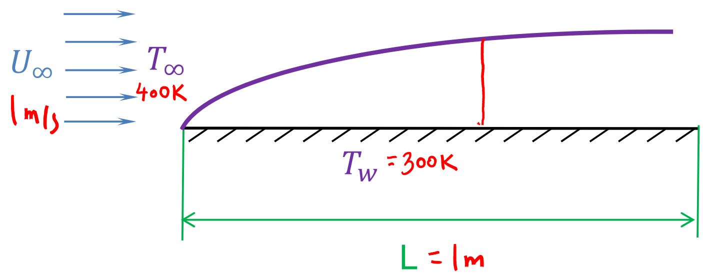


$$Re_L = \frac{\rho U_\infty L}{\mu} = 10{,}000$$

ρ = 1 $\frac{kg}{m^3}$, μ = 10^{-4} $\frac{Kg}{m\text{-}s}$

$$Pr = \frac{\mu C_p}{k} = 1$$

C_p = 10^4 $\frac{J}{Kg\,K}$, k = 1 $\frac{W}{m\,K}$

- Additional unknown variable: Temperature
- Need additional governing equation + BCs
- Does heating change the velocity and pressure field?

Governing Equations

Cornell Engineering

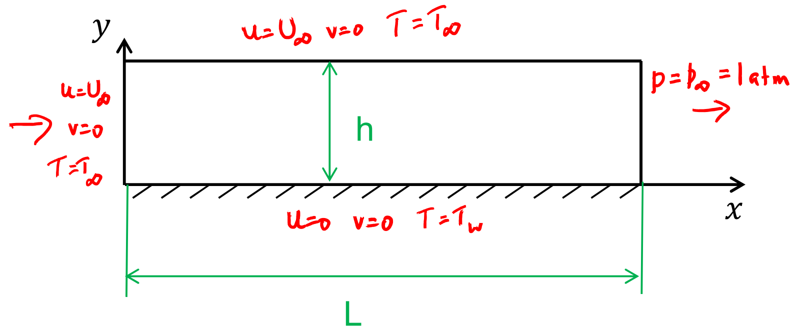
- Continuity
  - $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$
- $\vec{F} = m\,\vec{a}$ applied to a vanishingly small chunk of fluid
  - $\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$
  - $\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$
- Conservation of energy
  - $\rho C_p(\vec{V}\cdot\nabla)T = k\,\nabla^2 T + (\vec{V}\cdot\nabla)p + \Phi$

Assumptions:
2D, steady,
incompressible,
laminar,
Newtonian
Constant properties

Unknowns: $u(x,y), v(x,y), p(x,y)$, $T(x,y)$

Energy equation is uncoupled

Boundary Conditions



h is picked by user

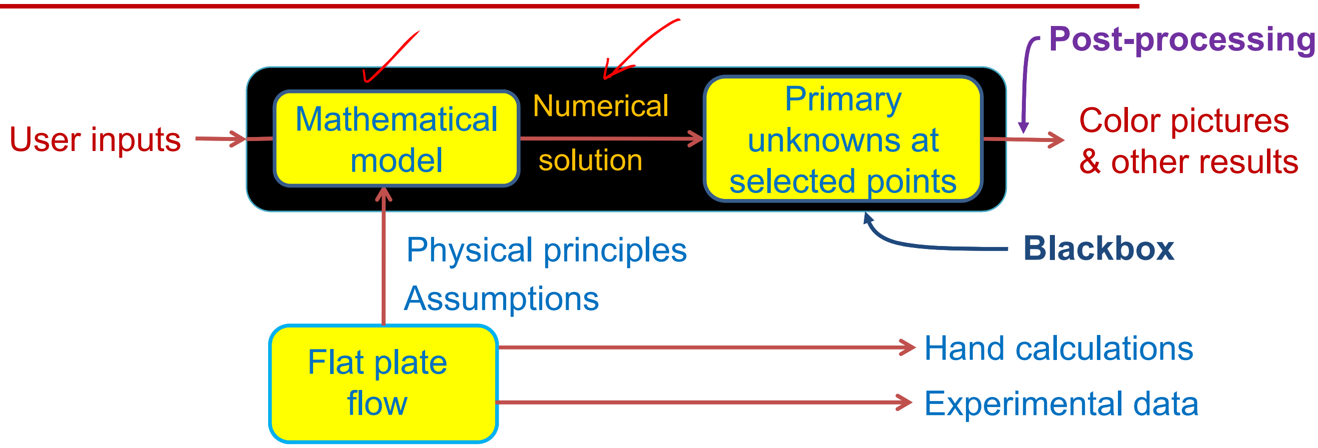
- Need to verify that choice of *h* doesn't affect the solution

Need to also check the effect of moving the left boundary

Investigating effect of outer boundaries is a basic check for all external flows

What's Inside the CFD Blackbox?

Cornell Engineering



How to Find Velocity and Pressure at Cell Centers?

Temp

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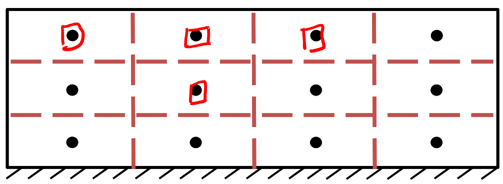
Mathematical Model (Boundary Value Problem) → System of algebraic equations in cell-center values → Invert → Cell-center values of u, v, p, T

Control volume balance for each cell

Post processing

$T(x,y)$ $p(x,y), u(x,y)$ Wall shear etc.

Each algebraic equation will relate a cell-center value to its neighbors



Numerical Solution Strategy: Overview

Cornell Engineering

BVP

Differential form of governing equations + BCs → Integral form of governing equations + BCs → Discretization error → Algebraic equations relating cell-center values

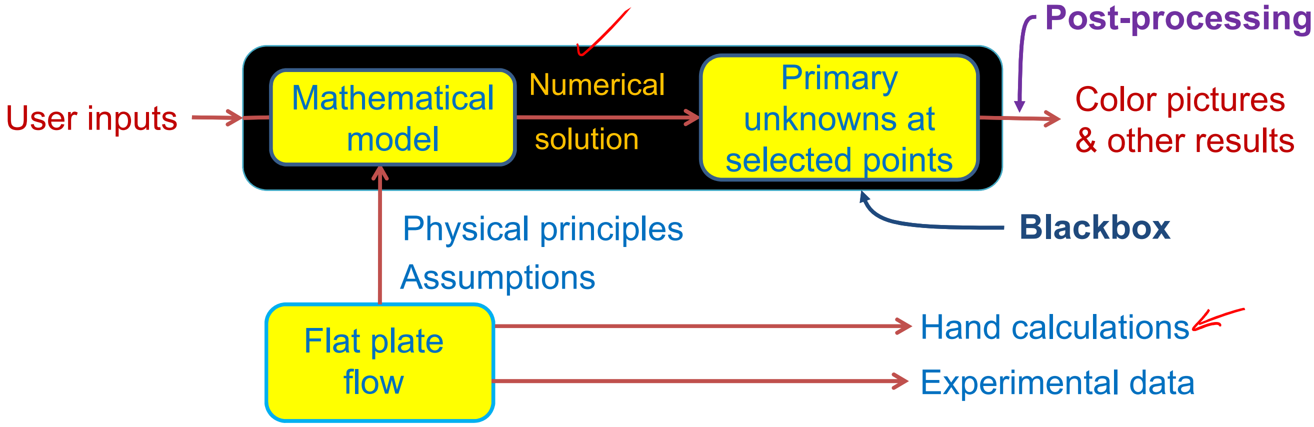
Linearization error

Linearized algebraic equations relating cell-center values → Solve iteratively updating guess after each iteration → Stop iterations when imbalances are below tolerance

For more details, see online lectures at https://goo.gl/cfiKHq

What's Inside the CFD Blackbox?

Cornell Engineering



Hand Calculations

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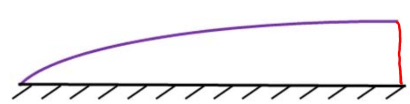
- For $\mathrm{Pr} = 1$,
  - Thermal boundary layer thickness is same as velocity boundary layer thickness
- Nusselt number
  - ➢ $Nu_x = \frac{hx}{k}$
  - ➢ $q_w = h\,(T_w - T_\infty)$
- Nusselt number correlation using boundary layer similarity solution
  - ➢ $Nu_x = 0.332\ Re_x^{1/2}\ Pr^{1/3}$
  - ➢ $Re_x = \frac{\rho\, U_\infty\, x}{\mu}$



$$\delta_{99\%}\Big|_{x=L} = 0.05\ \mathrm{m}$$

Post-processing

Cornell Engineering

- Non-dimensional temperature
  - $\frac{T_w - T}{T_w - T_\infty}$
- Nusselt number
  - $Nu_x = 0.332\ Re_x^{1/2}\ Pr^{1/3}$
  - $Nu_x = \frac{hx}{k}$
  - $q_w = h\,(T_w - T_\infty) = -k \left.\frac{\partial T}{\partial y}\right|_w$
- In Fluent,
  - $q_w = h_{eff}\,(T_w - T_{ref})$
- Plot Nu_x vs. Re_x
  - $Re_x = \frac{\rho\, U_\infty\, x}{\mu}$