Forced Convection Over a Flat Plate

Forced Convection in External Flows – Lesson 3

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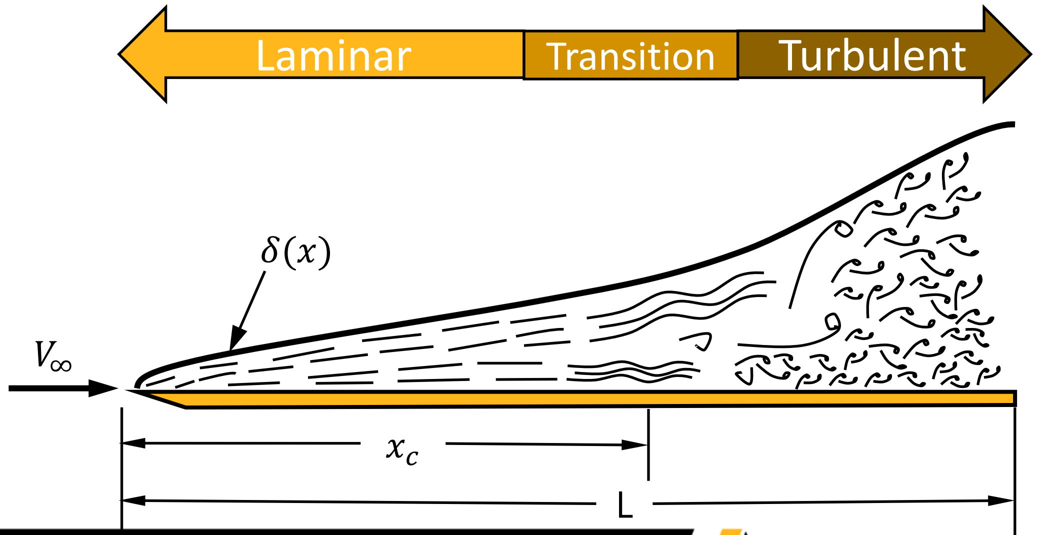
Laminar & Turbulent Flow Regimes

- The boundary layer flow over an infinitely long flat plate has three distinct regions based on the flow behavior:

Flow Regime	$Re_x = \frac{\rho_\infty V_\infty x}{\mu}$	Flow Characteristics
Laminar Flow Regime	$< 3.5 \times 10^5$ - 10^6	Smooth, highly ordered flow
Transitional Flow Regime*	$3.5 \times 10^5 - 10^6 < Re_x < \sim 3 \times 10^6$	Flow changes with time, sometimes laminar sometimes turbulent
Turbulent Flow Regime	$Re_x > \sim 3 \times 10^6$	Highly irregular random flow dominated by chaotic mixing

- The flow over a flat plate transitions from laminar to turbulent due to the interactions of naturally occurring unsteady flow structures or small disturbances within the boundary layers.
- The key to understanding and estimating convection is to identify the flow regime over the flat plate.
- We will investigate two thermal conditions:
  - Flat Plate with Isothermal Wall (constant wall temperature)
  - Flat Plate with Constant Heat Flux



The Re_x where transition occurs varies based on flow conditions, surface disturbances and flow application.

Hydrodynamic Boundary Layer: Blasius Solution (Laminar Flow)

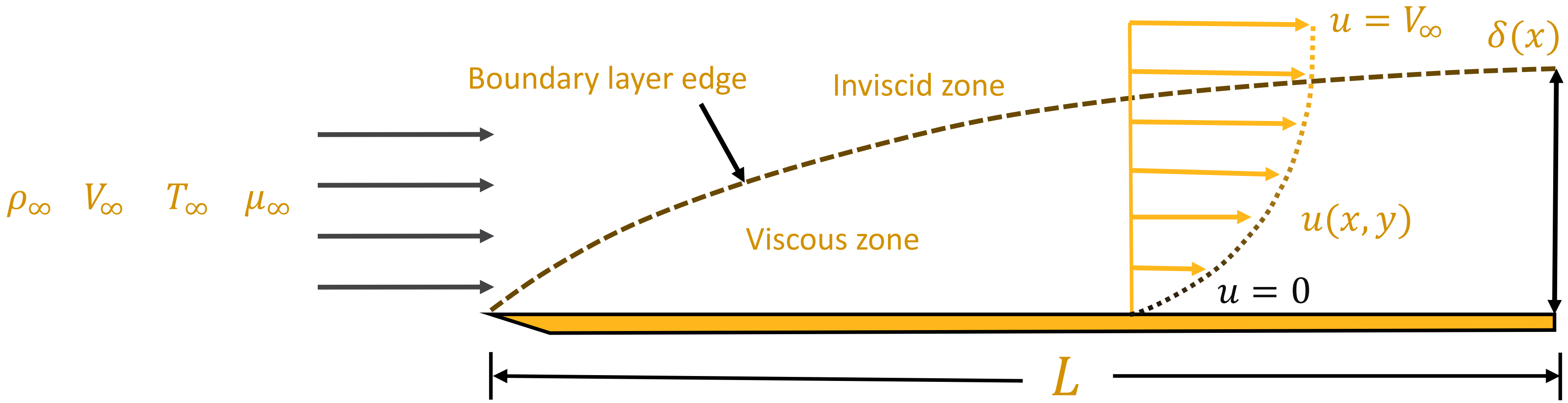
- Blasuis developed a similarity solution based on the observation that the velocity profile $\frac{u}{V_\infty}$ remains geometrically similar along the boundary layer.
- New independent and dependent variables η and $f(\eta)$ are defined in terms of stream function, and these variables are used to reduce the flat plate boundary layer partial differential equations (PDEs) into a single ordinary differential equation (ODE).
- The appropriate boundary conditions are also converted in terms of the similarity variables.
- A solution to the ODE is obtained by series expansion or numerical integration and the Hydrodynamic Boundary Layer profile is obtained.

$$\eta(x,y) \sim \frac{y}{\delta(x)} = y\sqrt{\frac{V_\infty}{\nu x}}$$

$$\psi = \sqrt{\nu V_\infty x} f(\eta)$$

$$f''' + \frac{1}{2} f f'' = 0$$

$$f(\eta = 0) = 0$$
$$f'(\eta = 0) = 0$$
$$f'(\eta \to \infty) = 1$$



$$\delta(x) \sim \frac{x}{\sqrt{Re_x}}$$

$$Re_x = \frac{\rho_\infty V_\infty x}{\mu_\infty}$$

Blasius Results (Laminar Flow)

- Hydrodynamic Boundary Layer Thickness: $$\frac{\delta(x)}{x} = \frac{5.0}{\sqrt{Re_x}}$$

- Wall shear stress is obtained from the velocity gradient at the wall $f''(0) = 0.332$:

$$\tau_w = \mu \frac{\partial u}{\partial y}\Big|_{y=0} = 0.332 \frac{\mu V_\infty}{\sqrt{\nu x / V_\infty}}$$

- The friction coefficient C_f is computed as follows:

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho_\infty V_\infty^2} = \frac{0.664}{\sqrt{Re_x}}$$

Laminar Flow over an Isothermal Flat Plate

- Let us assume a heated flat plate at a constant temperature, T_s.
- We define the dimensionless temperature, T^*

$$T^* = \frac{(T - T_s)}{(T_\infty - T_s)} \longrightarrow T^* = T(\eta)$$

Similarity Solution

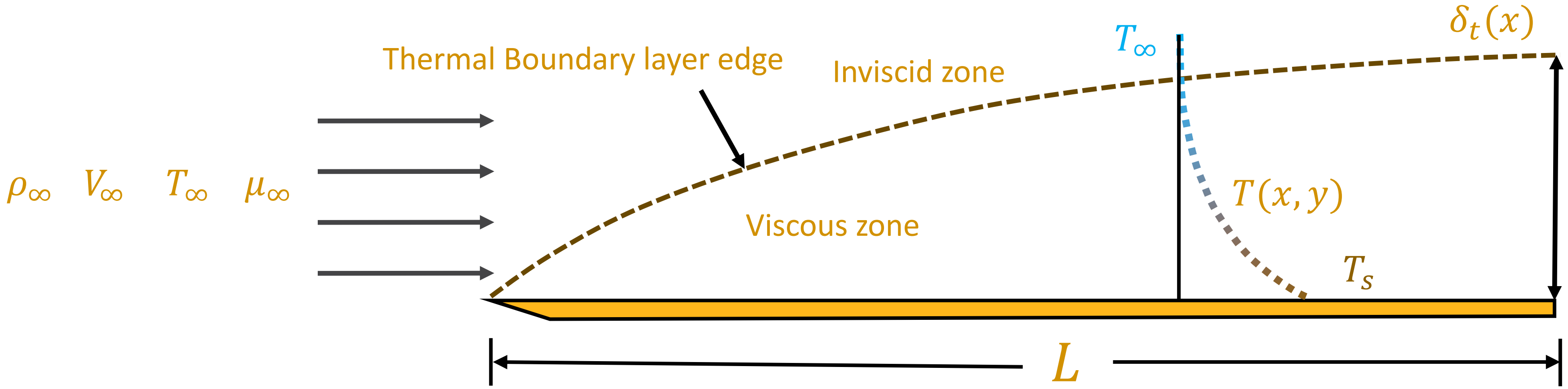
- The thermal boundary layer equation is transformed into the following ODE. This equation is dependent on the hydrodynamic conditions, $f(\eta)$.

$$\frac{d^2T^*}{d\eta^2} + \frac{Pr}{2} f(\eta) \frac{dT^*}{d\eta} = 0 \qquad Pr = \frac{\nu}{\alpha}$$

- The corresponding boundary conditions are: $T^*(0) = 0$ and $T^*(\infty) = 1$

At $y = 0, T = T_s$

At $y = \infty, T = T_\infty$



Thermal Boundary Layer Solution for Higher Prandtl Numbers

- For different values of Prandtl numbers, the ODE is numerically solved.
- For $Pr \geq 0.6$, Pohlhausen correlated the first derivative of T^* at $\eta = 0$ using the following correlation:

$$\frac{dT^*(0)}{d\eta} = 0.332 Pr^{\frac{1}{3}}$$

- Comparing the solutions of hydrodynamic and thermal boundary layer thicknesses, we obtain

$$\frac{\delta}{\delta_t} = Pr^{\frac{1}{3}}$$

$\Rightarrow$ For $Pr \gg 1, \delta \gg \delta_t$

Thermal BL is much thinner compared to the Hydrodynamic BL for fluids with Prandtl Numbers greater than 1.

- The local convective heat transfer coefficient is rewritten in terms of variables T^* and η:

$$h_x = \frac{q''_s}{(T_s - T_\infty)} = -\frac{(T_\infty - T_s)}{(T_s - T_\infty)} k \frac{\partial T^*(0)}{\partial y} = k \left(\frac{V_\infty}{\nu x}\right)^{0.5} \frac{dT^*(0)}{d\eta}$$

- The local Nusselt Number (at any x) is:

$$Nu_x = \frac{h_x x}{k} = 0.332 Re_x^{\frac{1}{2}} Pr^{\frac{1}{3}}$$

For $Pr \geq 0.6$

Average Nusselt Number for Laminar Flow

- The local Nusselt number expressions are obtained assuming an infinite flat plate.
- For the laminar region $(0 < x < x_c)$, the average heat transfer coefficient is determined by integrating the local heat transfer coefficient along the plate.
- For $Pr \geq 0.6$, the average heat transfer coefficient for laminar flow is:

$$\bar{h}_x = \frac{1}{x}\int_0^x h_x dx = 0.332 Pr^{\frac{1}{3}} \frac{k}{x}\left(\frac{V_\infty}{\nu}\right)^{\frac{1}{2}} \int_0^x \frac{dx}{x} = 2h_x$$

- The corresponding average Nusselt number for laminar flow over a flat plate is:

$$\overline{Nu}_x = \frac{\bar{h}_x x}{k} = 0.644 Re_x^{\frac{1}{2}} Pr^{\frac{1}{3}} = 2Nu_x$$

$Pr \geq 0.6$

Note: We can replace 'x' with 'L' if the flow is laminar over the entire flat plate.

For Laminar Flow over a Flat Plate, the Average Nusselt Number is twice the Local Nusselt Number.

Thermal Boundary Layer Solution for Low Prandtl Number

- For smaller Prandtl numbers (mainly liquid metals), the thermal boundary layer develops faster compared to the hydrodynamic boundary layer ($\delta_t \gg \delta$).
- In the limiting case $Pr \rightarrow 0$, because of the thin hydrodynamic boundary layer, we assume a uniform velocity $u = V_\infty$ throughout the thermal boundary layer. This allows us to set $f'(\eta) = 1$ in the thermal boundary layer equation.
- Differentiating the thermal boundary layer equation, we obtain:

$$\frac{d}{d\eta}\left(\frac{\frac{d^2T^*}{d\eta^2}}{\frac{dT^*}{d\eta}}\right) = -\frac{Pr}{2}\cancelto{1}{f'(\eta)}$$

- The solution of the above equation gives:

$$\frac{dT^*(0)}{d\eta} = \left(\frac{Pr}{\pi}\right)^{1/2}$$

$$Nu_x = \frac{h_x x}{k} = 0.565 Re_x^{\frac{1}{2}} Pr^{\frac{1}{2}} = 0.565 Pe_x^{\frac{1}{2}}$$

For $Pr \leq 0.5$

Peclet Number, $Pe_x = Re_x Pr$

- Even though the above local Nusselt number is for a limiting case of $Pr \rightarrow 0$, the result holds for $Pr \leq 0.5$

Turbulent Flow Over an Isothermal Flat Plate

- The flow Reynolds number (Re) for turbulence is generally greater than $\sim 3 \times 10^6$.
- Unlike laminar flow, we cannot use similarity variables to obtain the relationships for the turbulent flow regime. In such cases, we rely on experimental correlations to calculate the local coefficient of friction:

$$C_f = 0.0592 Re_x^{-\frac{1}{5}}$$

- In this flow regime, the heat transfer is dictated by random fluctuations as opposed to molecular diffusion, and therefore, the boundary layer growth is not related to the Prandtl number of the fluid.
- The thermal and hydrodynamic boundary layer thicknesses grow at a similar rate and their thickness is obtained from:

$$\delta \approx \delta_t = 0.37 x Re_x^{-\frac{1}{5}}$$

Turbulent boundary layer grows faster $\left(\delta \propto x^{\frac{4}{5}}\right)$ than laminar boundary layer $\left(\delta \propto x^{\frac{1}{2}}\right)$

- The local Nusselt number for turbulent flow is obtained by invoking the Chilton-Colburn analogy and using the above local coefficient of friction:

$$Nu_x = 0.0296 Re_x^{\frac{4}{5}} Pr^{\frac{1}{3}}$$

$0.6 \lesssim Pr \lesssim 60$

Mixed Boundary Layer Conditions

- Laminar relations are useful when the flow over the flat plate is either entirely laminar or marginally transitional quite close to the trailing edge of the plate (in the last ~5% of the plate length).

- If the transitional region occurs reasonably close to the leading edge of the flat plate, using laminar relations for predicting the convective heat transfer will produce erroneous results. The heat transfer in such cases is affected by both laminar and turbulent boundary layers.

- If the transition occurs at a length x_c (laminar region: $0 < x < x_c$ and turbulent region: $x_c < x < L$), it is useful to estimate the average heat transfer coefficient over the entire plate as:

$$\bar{h}_L = \frac{1}{L}\left(\int_0^{x_c} h_{lam} dx + \int_{x_c}^{L} h_{turb}\, dx\right)$$

- Substituting for the local laminar and turbulent heat transfer coefficients and integrating, we obtain the following relationship for the average Nusselt number.

$$\overline{Nu}_x = \left(0.037 Re_L^{\frac{4}{5}} - A\right) Pr^{\frac{1}{3}}$$

$0.6 \lesssim Pr \lesssim 60$

$Re_{x,c} \lesssim Re_L \lesssim 10^8$

where, A is a constant estimated from the critical Reynolds Number, $Re_{x,c}$

$$A = 0.037 Re_{x,c}^{\frac{4}{5}} - 0.664 Re_{x,c}^{\frac{1}{2}}$$

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Laminar & Turbulent Flow: Constant Heat Flux

- In certain heating and cooling applications, such as electronics and food processing industries, the wall heat flux is known. For such applications, the local wall temperature distribution becomes the key variable of interest.
- The local Nusselt number relations for a flat plate with a constant heat flux $\left(q''_s\right)$ are:

Laminar Flow: $$Nu_x = 0.453 Re_x^{\frac{1}{2}} Pr^{\frac{1}{3}}$$ $Pr \geq 0.6$

Turbulent Flow: $$Nu_x = 0.0308 Re_x^{\frac{4}{5}} Pr^{\frac{1}{3}}$$ $0.6 \lesssim Pr \lesssim 60$

- Comparing these with the isothermal relations:
  - Laminar Flow: The Nusselt number is ~36% higher for the constant heat flux condition.
  - Turbulent Flow: The Nusselt number is ~4% higher for the constant heat flux condition.
- The wall temperature is determined using the local heat transfer coefficient in the following expression:

$$T_w(x) = T_\infty + \frac{q''_s}{h_x}$$

Summary

- For a flat plate under isothermal conditions,
  - Laminar flow:
    - For high Prandtl numbers $(Pr \gg 1)$, the hydrodynamic BL is thicker than the thermal BL.
    - For low Prandtl numbers $(Pr \leq 0.5)$, the thermal BL is thicker than the hydrodynamic BL.
  - Turbulent Flow:
    - The thermal and hydrodynamic BL thickness are nearly equal, and the BL growth is independent of the Prandtl number.
    - Heat transfer is primarily governed by random fluctuations (as opposed to molecular diffusion).
- We discussed relationships for local and average Nusselt numbers relations for flat plates in both isothermal and uniform heat flux conditions.
- In the transitional regime, it is useful to estimate the average heat transfer coefficient from both laminar and turbulent relationships.

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