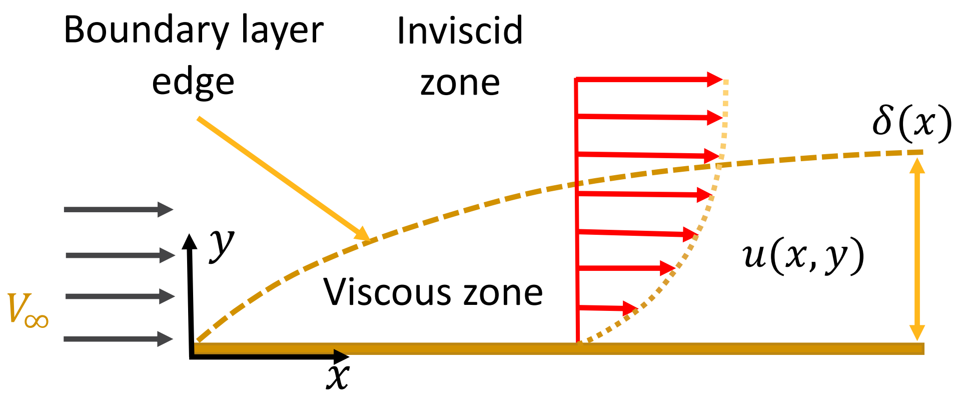
Velocity & Thermal Boundary Layers

Forced Convection in External Flows – Lesson 2

Velocity and Thermal Boundary Layers

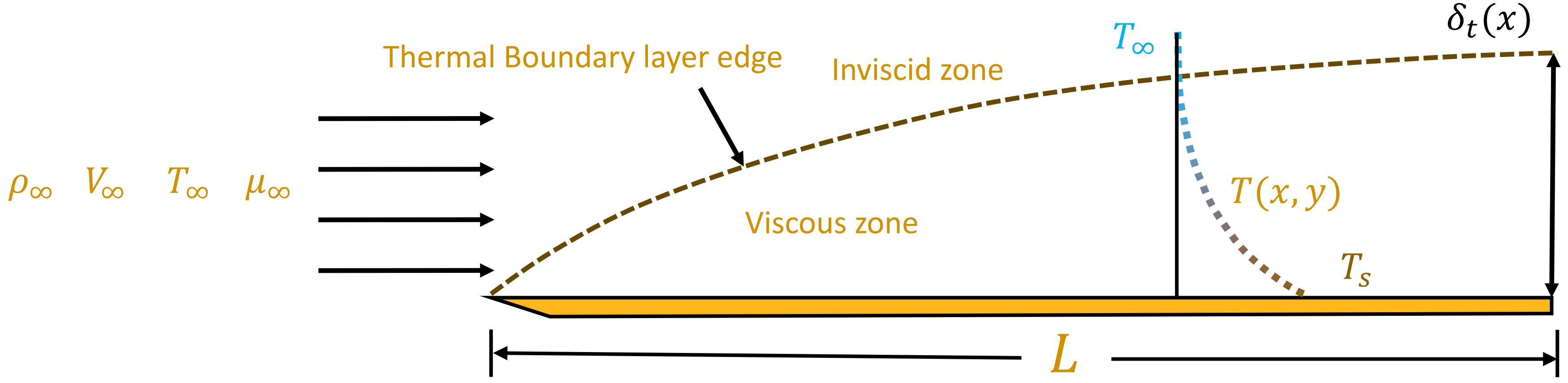
- The problem of determining local or averaged HTCs is complex, as they depend on various fluid properties and the surface geometry.
- This dependency is a result of the convection heat transfer being determined by the boundary layers along the surface.
- Thus, it is important to give a quick review of the velocity boundary layer concept and introduce its thermal counterpart.
- The concept of the velocity boundary layer was introduced by Prandtl in 1904. The boundary layer is the thin region next to the wall where viscous effects dominate.
- As a result of the no-slip condition, a sharp gradient of velocity exists next to the wall, resulting in skin friction drag and other effects.
- Within the boundary layer, the continuity and momentum equations can be reduced to simpler forms, which permit solutions for simple geometries (e.g., flat plates).



Steady-state velocity boundary layer model for flow over a flat plate

Thermal Boundary Layers

- The idea behind velocity boundary layers can be extended to thermal problems .
- In an analogous fashion, sharp gradients of temperature are observed in a layer of fluid next to a wall boundary in a viscous flow. The thickness of this layer is denoted $\delta_t(x)$.
- The temperature gradients result in heat fluxes (Fourier's Law) and, therefore, heat transfer between the fluid and the wall.



Steady-state thermal boundary layer model for flow over a flat plate

Laminar Boundary Layer Approximations

- The 2D governing equations can be simplified by applying well-known boundary layer approximations:

$$u \gg v \qquad \frac{\partial u}{\partial y} \gg \frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial v}{\partial x}$$

$$\frac{\partial T}{\partial y} \gg \frac{\partial T}{\partial x}$$

- Under these assumptions, the steady incompressible boundary layer equations for laminar flows are:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

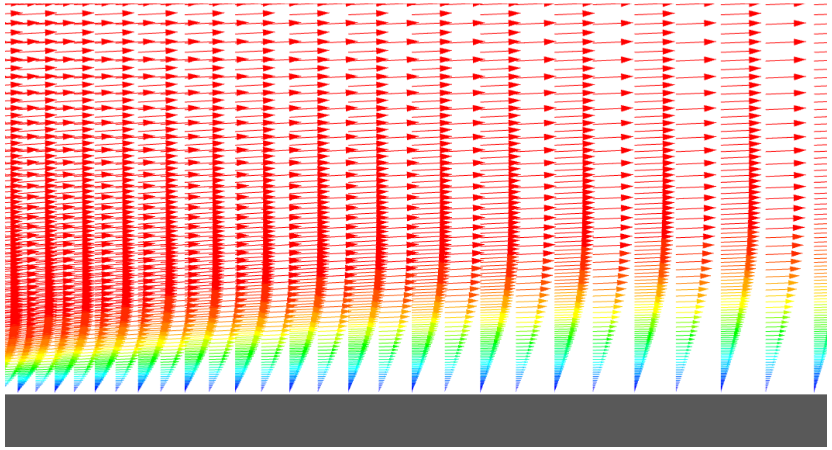
$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} + \underbrace{\frac{\nu}{c_p}\left(\frac{\partial u}{\partial y}\right)^2}_{\text{viscous dissipation}}$$

- The viscous dissipation term is typically small compared to advection terms and can be neglected in most incompressible flows.

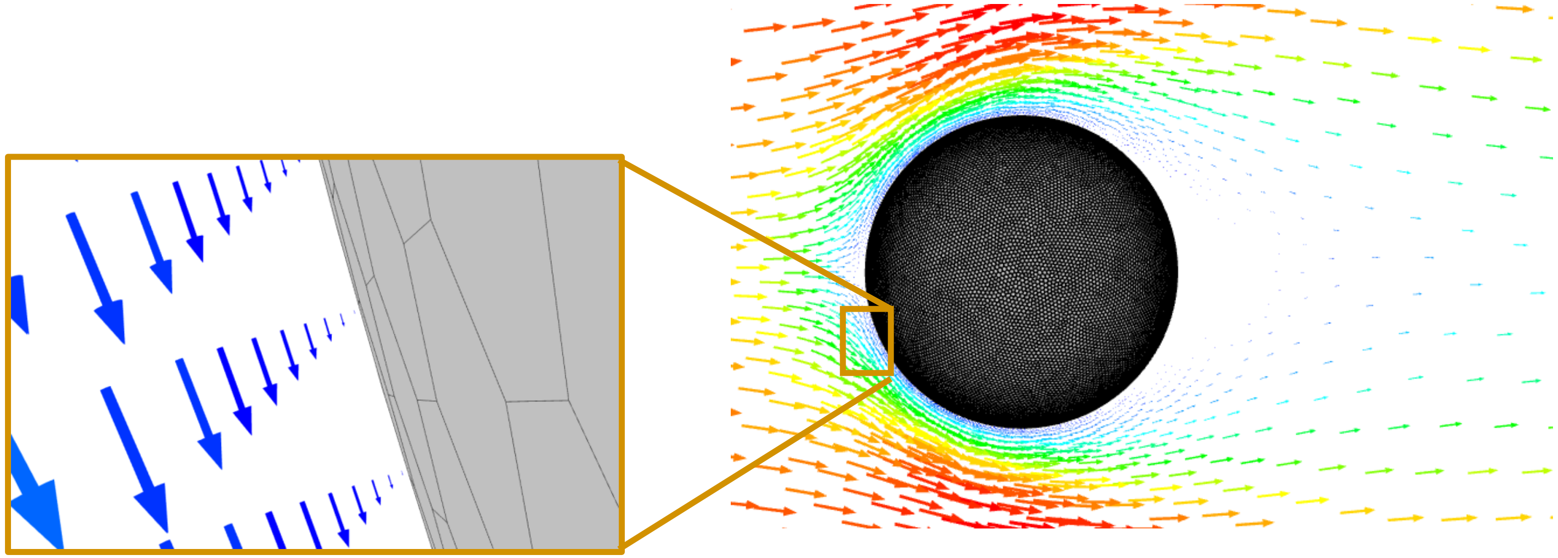
Laminar Boundary Layer Approximations (cont.)

- For constant property flows, the continuity and momentum equations are decoupled from the energy equation, and the boundary layer velocity field can be calculated without knowledge of the temperature.
- Temperature field, however, depends on the velocity, and velocities must be known before the equation for temperature is solved.
- One can utilize well-known relations and solutions for the velocity boundary layer to derive corresponding thermal layer relations.



Laminar Boundary Layer over a Flat Plate

Boundary Layer over a Sphere



Non-Dimensional Boundary Layer Equations

- The non-dimensional form of the boundary layer equations is obtained by normalizing independent and dependent variables by characteristic length, velocity and temperature scales:

$$x^* \equiv \frac{x}{L}, \qquad y^* \equiv \frac{y}{L}, \qquad u^* \equiv \frac{u}{V_\infty}, \qquad v^* \equiv \frac{v}{V_\infty}, \qquad T^* \equiv \frac{T - T_w}{T_\infty - T_w}$$

$$\Downarrow$$

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial u^*}{\partial y^*} = 0$$

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = -\frac{dp^*}{dx^*} + \frac{1}{Re_L}\left(\frac{\partial^2 u^*}{\partial y^{*2}}\right)$$

$$u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*} = \frac{1}{Re_L Pr}\left(\frac{\partial^2 T^*}{\partial y^{*2}}\right) + \frac{Br}{Re_L Pr}\left(\frac{\partial u^*}{\partial y^*}\right)^2$$

Usually small and can be neglected

- Here Re_L, Pr and Br are non-dimensional numbers:

Reynolds Number

$$Re_L = \frac{\rho V L}{\mu}$$

Prandtl Number

$$Pr = \frac{c_p \mu}{k} = \frac{\nu}{\alpha}$$

Brinkman Number

$$Br = \frac{\mu V_\infty^2}{k(T_\infty - T_w)}$$

The product of Reynolds and Prandtl numbers in the thermal diffusion term is sometimes defined as the Peclet Number, Pe_L.

Physical Interpretation of Non-Dimensional Parameters

- The value of the Prandtl number reflects the relative growth of velocity and thermal boundary layers. For laminar flows,

$$\frac{\delta}{\delta_t} \approx Pr^n$$

- For gases, $Pr \sim 1$, and velocity and thermal boundary layers have approximately the same thicknesses.
- For liquids, $Pr \gg 1$, and the thermal layer is much thinner than the velocity layer.
- For liquid metals, $Pr \ll 1$, and the velocity layer is much thinner than the thermal layer.
- This observation has important implications in numerical simulations of convective heat transfer, as thermal laminar layers in liquids and liquid metals require much different near-wall resolution than the velocity layers, while no special considerations for resolving thermal layers are needed in gas flows.

Boundary Conditions

- Viscous flow problems with the boundary layer equations can be solved by:
  1. Computing the pressure field dp/dx around the body using inviscid methods (e.g., potential flow).
  2. Computing the viscous flow field near the walls using the boundary layer equations.
- The addition of the energy equation requires:
  1. A thermal wall boundary condition (specified temperature or heat flux).
  2. Freestream conditions for the temperature outside the boundary layer.
- Thus, for the boundary layer equations (including the energy equation), the boundary conditions become:

$$u^*(x,0) = 0$$
$$v^*(x,0) = 0$$

No-slip BC

$$T^*(x,0) = T_W \text{ or } \frac{\partial T^*}{\partial y^*} = \frac{q''L}{T_\infty k_\infty}$$

Thermal wall BC

$$u(x,y) \rightarrow V_\infty \quad as\, y \rightarrow \infty$$
$$T(x,y) \rightarrow T_\infty \quad as\, y \rightarrow \infty$$

Freestream conditions at boundary layer edge

Reynolds Analogy

- It can be noted from the governing boundary layer equations that when $Pr = 1$ and the pressure gradient is zero, the momentum and energy equations have identical non-dimensional form. This is a form of the Reynolds Analogy stating that the velocity and temperature profiles have the same shape.

- While, strictly speaking, this is valid for $Pr = 1$ and zero-pressure gradient, it is also a good approximation for gases in both laminar and turbulent flows.

- The Reynolds analogy allows us to relate wall heat flux (in terms of non-dimensional Nusselt number) and skin friction (in terms of a non-dimensional skin friction coefficient):

$$C_f \frac{Re_L}{2} = Nu$$

where $C_f = \frac{2}{Re_L} \frac{\partial u^*}{\partial y^*}\Big|_w$

- The significance of this relationship lies in the fact that, for a known velocity field, it provides an estimate for the Nusselt number (or heat transfer coefficient) for the flow.

Reynolds Analogy (cont.)

- The Reynolds analogy can be rewritten in terms of the Stanton number, St, as,

$$\frac{C_f}{2} = St \quad \text{where} \quad St = \frac{h}{\rho V_\infty c_p} = \frac{Nu}{RePr}$$

- The Reynolds analogy can be corrected by a Pr correction to expand its applicability to a wide range of Prandtl numbers. A popular form is the Chilton-Colburn analogy:

$$C_f/2 = StPr^{2/3} = j_H, \qquad 0.6 < Pr < 60$$

j_H - Colburn j factor

- For laminar flows, this analogy is appropriate only if:

$$\frac{dp^*}{dx^*} \cong 0$$

Turbulent Boundary Layer Equations

- A laminar boundary layer along a flat plate transitions to the turbulent regime at $Re_{x,c} = 3.5 \times 10^5 \ \ to\ \ 10^6$.
- Governing equations for a turbulent boundary layer can be derived by representing a flow variable (ϕ) as a sum of its mean ($\bar{\phi}$) and instantaneous fluctuating components (ϕ'), $\phi = \bar{\phi} + \phi'$, and applying the classical Reynolds averaging technique. For an incompressible flow with constant properties, the turbulent boundary layer equations are:

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0$$

$$\rho\left(\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y}\right) = -\frac{\partial \bar{p}}{\partial x} + \frac{\partial}{\partial y}\left(\mu\frac{\partial \bar{u}}{\partial y} - \rho\overline{u'v'}\right)$$

Reynold's stress term

$$\frac{\partial \bar{p}}{\partial y} = -\frac{\partial \overline{v'^2}}{\partial y}$$

This term is assumed to be small across the boundary layer (~0.4% of freestream dynamic pressure)

$$\rho c_p\left(\bar{u}\frac{\partial \bar{T}}{\partial x} + \bar{v}\frac{\partial \bar{T}}{\partial y}\right) = \frac{\partial}{\partial y}\left(k\frac{\partial \bar{T}}{\partial y} - \rho c_p\overline{v'T'}\right)$$

Turbulent Boundary Layer Equations (cont.)

- Extra terms appearing in the momentum and energy equations, $\overline{u'v'}$ and $\overline{v'T'}$, represent contributions of turbulent mixing to the total shear stress and total heat flux:

$$\tau_{tot} = \left(\mu \frac{\partial \bar{u}}{\partial y} - \rho \overline{u'v'}\right)$$

$$q''_{tot} = -\left(k \frac{\partial \bar{T}}{\partial y} - \rho c_p \overline{v'T'}\right)$$

- These expressions confirm that transport rates of momentum and heat are enhanced by the presence of turbulence.
- $\overline{u'v'}$ and $\overline{v'T'}$ are additional unknowns which require additional modeling.

Turbulent Boundary Layer Equations (cont.)

- The most common (but not exclusive) approach to modeling these terms is the Boussinesq hypothesis, which relates the Reynolds stress to gradients of mean flow velocity and temperature through an eddy (or turbulent) viscosity, $\mu_T = \rho\nu_T$, and eddy diffusivity, α_T:

$$\rho\nu_T \frac{\partial \bar{u}}{\partial y} = -\rho\overline{u'v'} \Rightarrow \tau_{tot} = \rho(\nu + \nu_T)\frac{\partial \bar{u}}{\partial y}$$

$$\alpha_T \frac{\partial \bar{T}}{\partial y} = -\overline{v'T'} \Rightarrow q''_{tot} = -\rho c_p(\alpha + \alpha_T)\frac{\partial \bar{T}}{\partial y}$$

Substituting in the BL equations

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0$$

$$\left(\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y}\right) = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} + \frac{\partial}{\partial y}\left((\nu + \nu_T)\frac{\partial \bar{u}}{\partial y}\right)$$

$$\left(\bar{u}\frac{\partial \bar{T}}{\partial x} + \bar{v}\frac{\partial \bar{T}}{\partial y}\right) = \frac{\partial}{\partial y}\left((\alpha + \alpha_T)\frac{\partial \bar{T}}{\partial y}\right)$$

Turbulent Boundary Layer Equations

- Methodologies of determining ν_T and α_T are defined by the science of Turbulence Modeling.

- In general, there is no universal turbulence model which is applicable to all types of turbulent flows, and the choice of a turbulence modeling approach depends on specific flow physics and the application.

- Even though the turbulent boundary layer equation resembles the laminar one, we rely on experiments as well as CFD to obtain the solution.

Summary

- We learned about the thermal boundary layer region and understood the general nature of the temperature profile.

- We simplified the 2D governing equations to obtain the final form of the laminar boundary layer equations. We also non-dimensionalized these equations. The non-dimensional solution helped in understanding the relative thickness between velocity and thermal boundary layers for different Prandtl numbers.

- For $Pr = 1$ and zero pressure gradients along the flow, we discussed the Reynolds Analogy and developed a relationship between the Skin Friction coefficient and the Nusselt number.

- We also outlined the turbulent boundary layer equations and defined the additional terms eddy (or turbulent) viscosity and eddy diffusivity that appear because of the Boussinesq approximation.

- Next, we will look at forced convection in flow over a flat plate.

Ansys