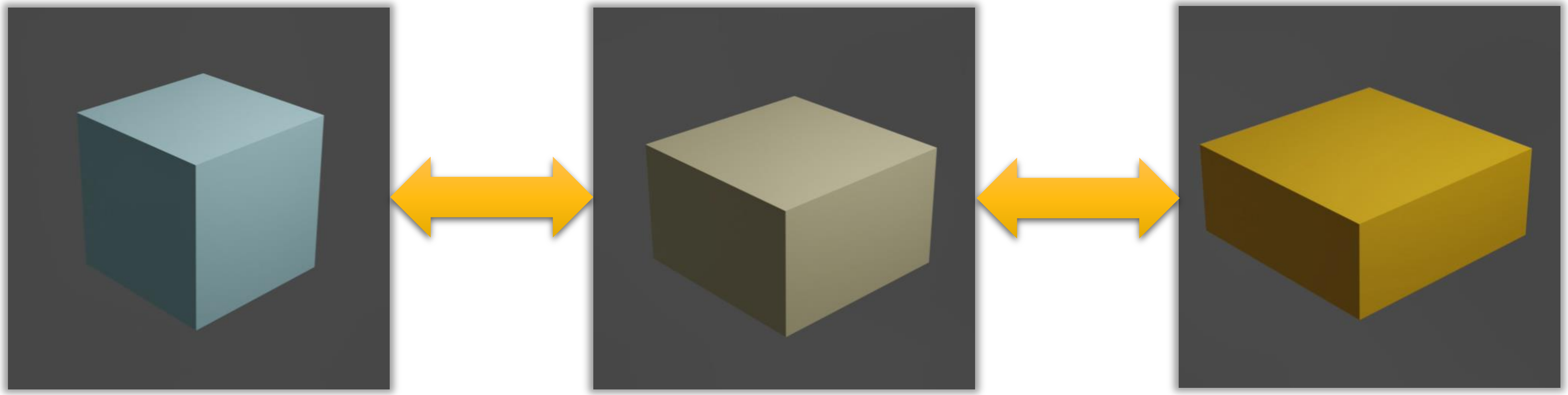
Energy Functional

Hyperelasticity – Lesson 2

Energy Functional

1. They undergo large deformations without sustaining permanent deformation.
2. All work done is stored as internal energy and is recovered upon unloading.
3. Key: all the work done is recoverable and process is fully reversible.



Follows Second law of thermodynamics!

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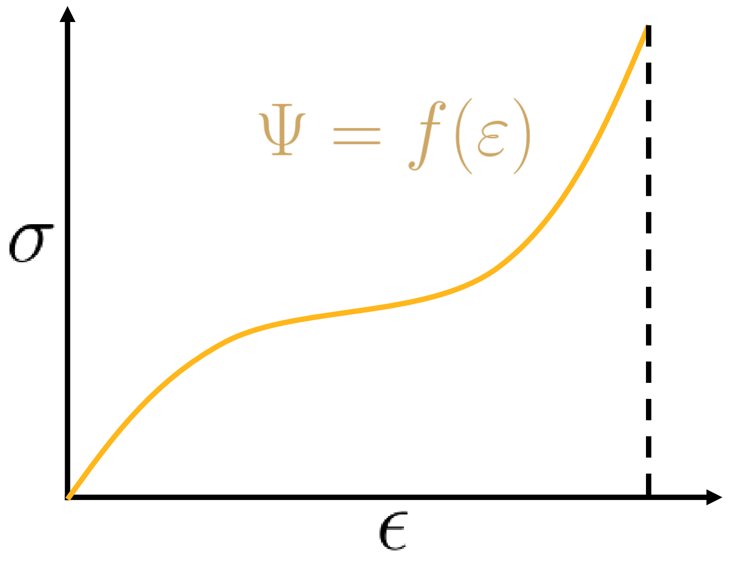
Hyperelastic Model

- Area under the curve is the internal or strain energy stored in the unit volume of the material.
- It remains constant during loading and unloading so it is used for modeling hyperelastic behavior.
- It is defined as a function of strain tensor.
- Stress developed in the part is calculated from the strain energy function.
- Strain energy function is usually expressed as additive split of deviatoric and volumetric energies.



$$\Psi = \Psi_{dev} + \Psi_{vol}$$

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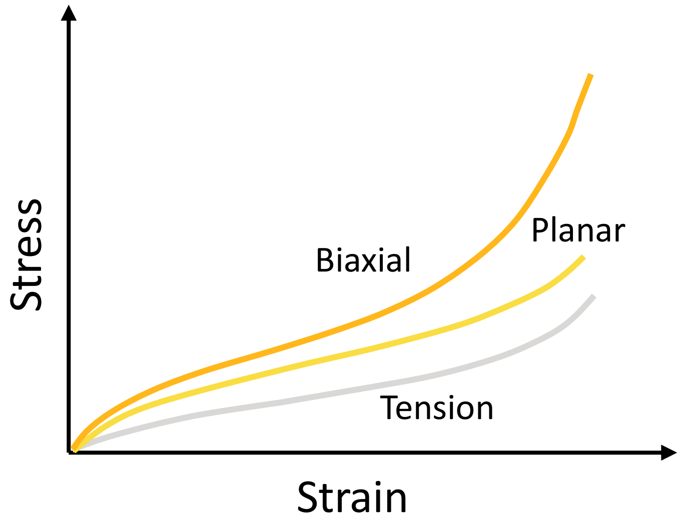
Energy Functional

- Stress developed depends on the type of strain.
- There are 3 important modes of deformation
  1. Uniaxial tension
  2. Uniaxial compression
  3. Shear
- How do we differentiate the type of strain while calculating stress?
- There are two ways to do that.



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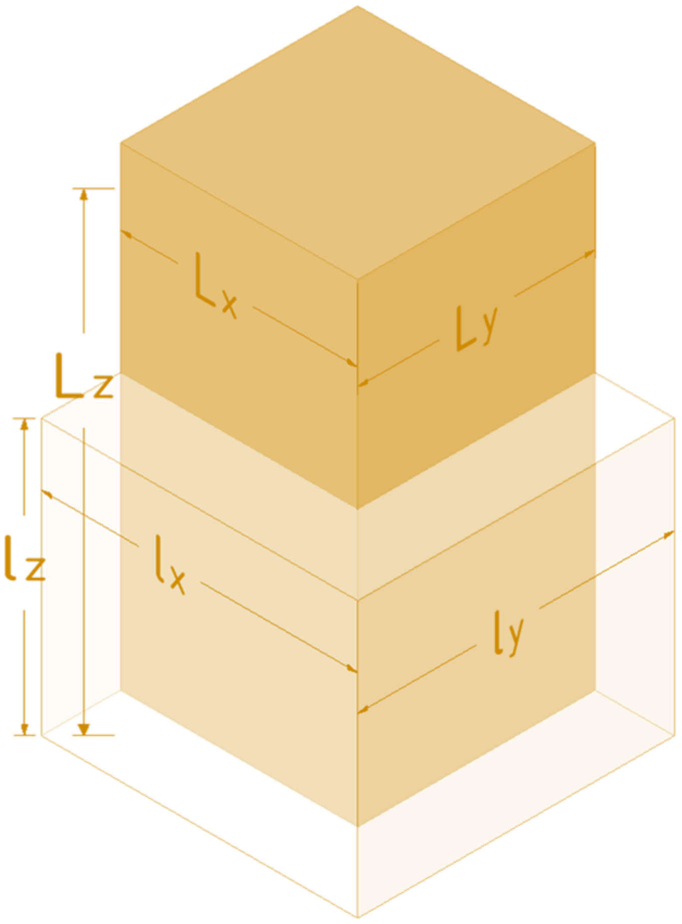
Approach I: Principal Stretches

- Let's consider a cube of side L stretching in one direction.
- Due to Poisson's effect the sides in other lateral directions change.
- Ratio of final length to initial length are principal stretches.

$$\lambda_x = \frac{l_x}{L_x} \qquad \lambda_y = \frac{l_y}{L_y} \qquad \lambda_z = \frac{l_z}{L_z}$$

- Stretch of 1 stands for undeformed state; a stretch value between 0 and 1 stands for compression and a value of stretch >1 stands for tension.



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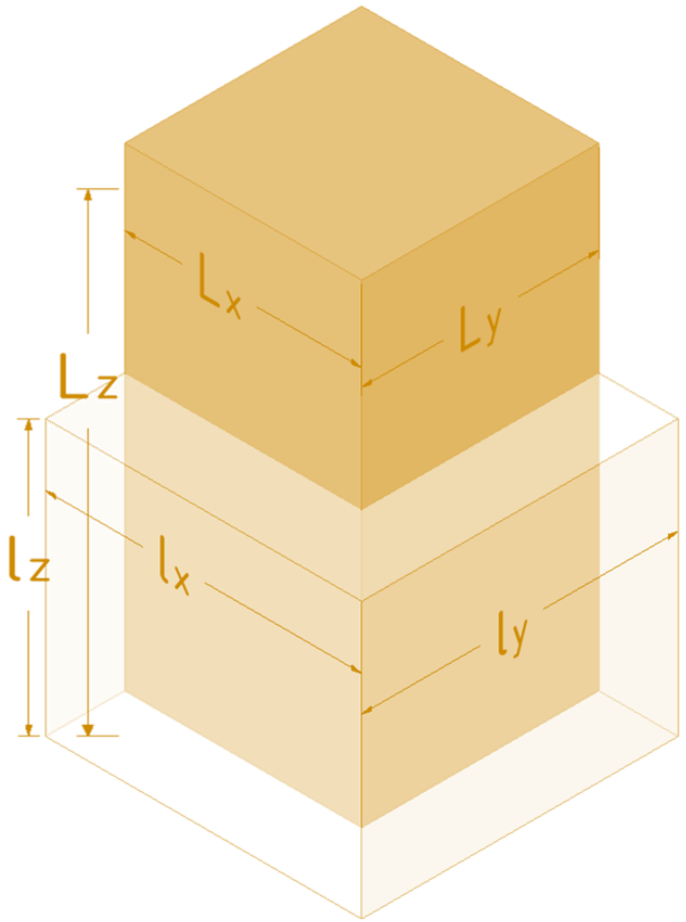
- If the cube is stretched by Δy in Y-direction, then

$$\lambda_y = \frac{L_y + \Delta y}{L_y} = 1 + \frac{\Delta y}{L_y} = 1 + e$$

- Where e is nothing but engineering strain.
- This way the principal stretch can be related to the engineering strain that is discussed in earlier module.



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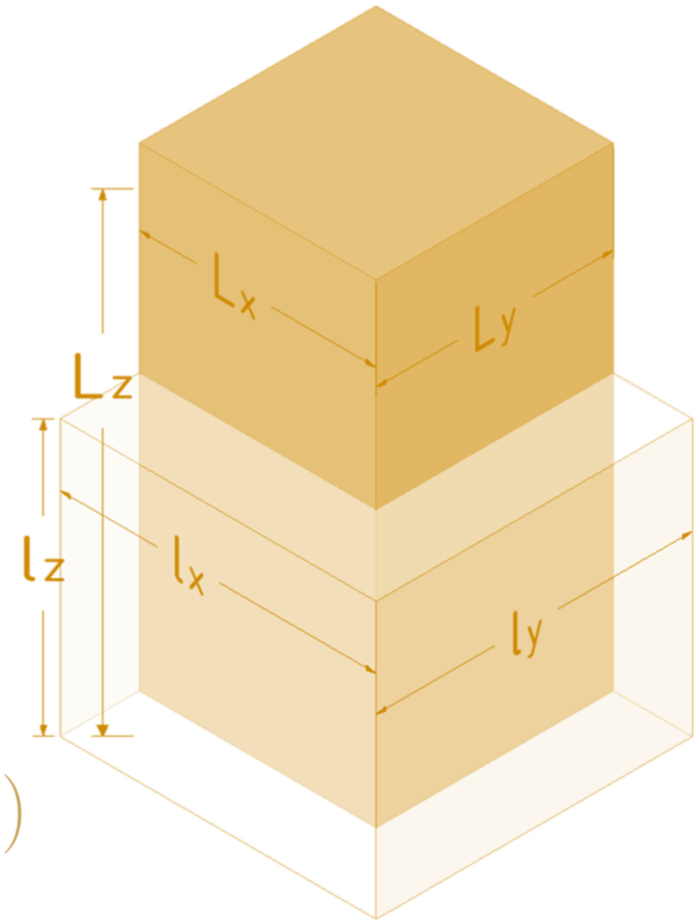
Approach II: Strain Invariants

- Another way of representing strain tensor is using strain invariants.
- Strain invariants do not change if the reference coordinate system changes.
- Strain invariants will remain the same for all modes of deformation which makes them ideal for defining strain energy function.
- Three invariants that are commonly used are I_1, I_2 & I_3.



$$I_1 = \mathrm{tr}(\varepsilon) \qquad I_2 = \frac{1}{2}\{\mathrm{tr}(\varepsilon^2) - [\mathrm{tr}(\varepsilon)]^2\} \qquad I_3 = \det(\varepsilon)$$

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- Let's discuss more about the third strain invariant, I_3.
- Using simple math it can be shown that I_3 is related to a quantity called Jacobian, which is nothing but determinant of deformation gradient.

$$I_3 = J^2$$

- The Jacobian is ratio of the deformed to the undeformed infinitesimal volume elements.

$$J = \det(\mathbf{F}) = \frac{V}{V_0}$$

- In other words, it is a measure of change in volume of the element due to deformation.
- Therefore, it is used for defining the volumetric component of energy function.

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Model Formulation

- Energy functional is defined in terms of either principal stretches or the strain invariants.

$$\Psi = f(I_1, I_2, I_3) \qquad \Psi = f(\lambda_1, \lambda_2, \lambda_3)$$

- Energy function is also referred to as strain energy density function.
- Stresses in the material are calculated from this function using second law of thermodynamics.

$$\mathbf{S} = 2\frac{\partial \Psi}{\partial \mathbf{C}}$$

- Where S is the second Piola-Kirchoff stress.
- Let's look at some commonly used formulations.

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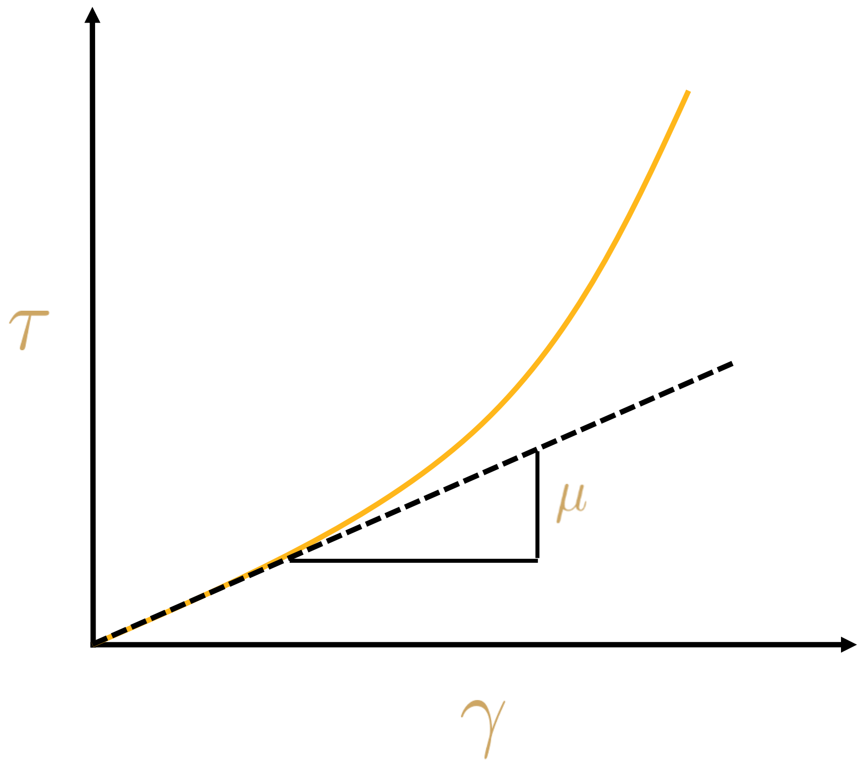
Neo-Hookean Model

- Neo-Hookean is the simplest hyperelastic material model.
- It is defined using strain invariants
- It has two material parameters: μ and d.

$$\Psi = \frac{\mu}{2}(I_1 - 3) + \frac{1}{d}(J - 1)^2$$

- Physical meaning of parameters:
- μ is the initial shear modulus.
- d is 2/(initial bulk modulus).



Note that shear and bulk modulus change with strain.

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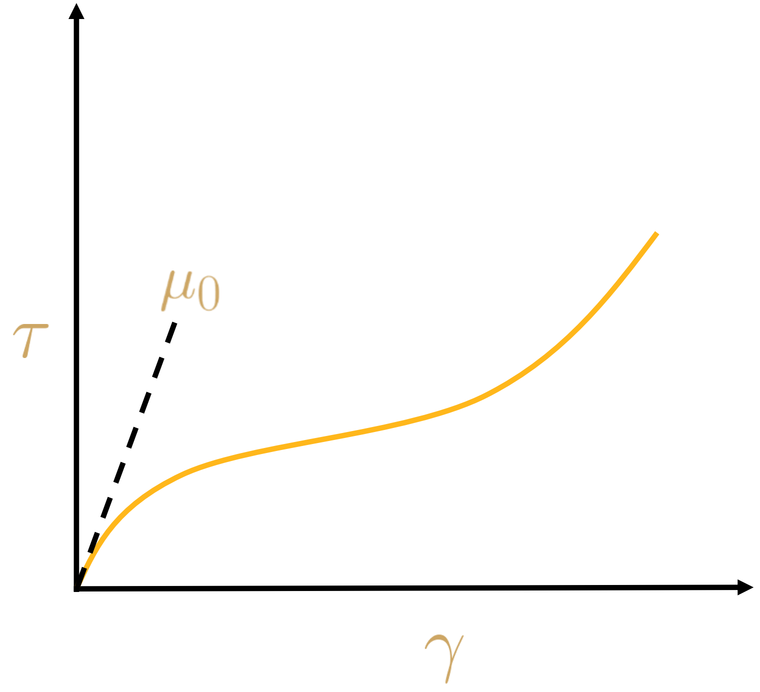
Mooney-Rivlin Model

- Mooney-Rivlin is another commonly used model which also uses strain invariants.
- It has 3 material properties: C10, C01, & d.

$$\Psi = C_{10}(I_1 - 3) + C_{01}(I_2 - 3) + \frac{1}{d}(J - 1)^2$$

- Using multiple terms make it more nonlinear so it is better suited to model highly nonlinear behavior.
- Physical meaning of parameters:
  1. Initial shear modulus, $\mu_0 = 2(C_{10}+C_{01})$
  2. Initial bulk modulus, $K_0 = 2/d$



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Ogden Model

- Ogden form is another commonly used hyperelastic model.
- It uses principal stretches for defining energy functional.

$$\Psi = \sum_{i=1}^{N} \frac{\mu_i}{\alpha_i}(\lambda_1^{\alpha_i} + \lambda_2^{\alpha_i} + \lambda_3^{\alpha_i} - 3) + \sum_{k=1}^{N} \frac{1}{d}(J-1)^{2k}$$

- Both deviatoric and volumetric parts can have multiple terms.
- Each deviatoric term has 2 material constants and each volumetric term has 1 material constant.
- Physical meaning of material constants:
  1. Initial shear modulus, $\mu_0 = \frac{1}{2}\sum_{i=1}^{N}(\alpha_i \mu_i)$
  2. Initial bulk modulus, $K_0 = \sum_{k=1}^{N} \frac{2}{d_k}$
  3. α is dimensionless, nonlinearity constant.

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Ogden Foam Model

- Ogden compressible foam model is similar to Ogden form with few modifications to account for compressible behavior.

$$\psi = \sum_{i=1}^{N} \frac{\mu_i}{\alpha_i}\left(J^{\frac{\alpha_i}{3}}\left[\bar{\lambda}_1^{\alpha_i} + \bar{\lambda}_2^{\alpha_i} + \bar{\lambda}_3^{\alpha_i}\right] - 3\right) + \sum_{i=1}^{N} \frac{\mu_i}{\alpha_i \beta_i}\left[J_1^{-\alpha_i \beta_i} - 1\right]$$

- Where initial shear and bulk modulus are

$$\mu_0 = \frac{\sum_{i=1}^{N} \mu_i \alpha_i}{2} \qquad K_0 = \sum_{i=1}^{N} \mu_i \alpha_i \left(\frac{1}{3} + \beta_i\right)$$

- In this model the deviatoric and volumetric terms are tightly coupled.
- This model is used for highly compressible behavior.

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Other Energy Functions

1. Polynomial form
2. Yeoh hyperelasticity
3. Gent hyperelasticity
4. Arruda-Boyce hyperelasticity
5. Blatz-ko
6. Etc.

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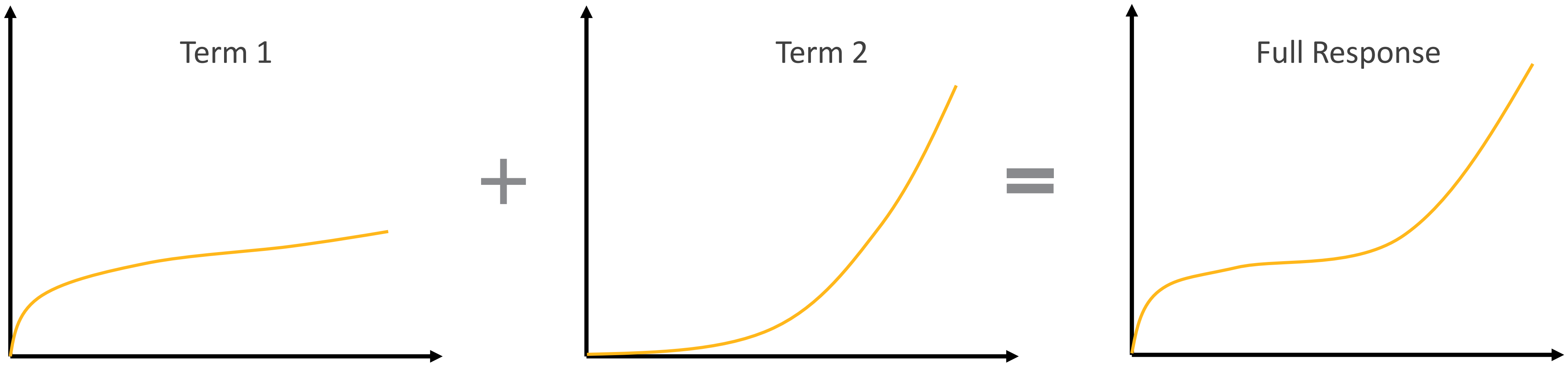
A Note About Multi-term Models

- We have seen a few models that have multiple terms (e.g., Ogden, Ogden foam, Mooney-Rivlin, etc.,).
- Multiple terms increase the nonlinearity of the model and therefor are useful in capturing highly nonlinear behavior.
- Phenomenologically, they represent multiple chains in the microstructure with each chain type responsible for each term.



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Incompressible vs Compressible Behavior

- In the preceding slides we have seen some forms of energy functions that are suitable for incompressible and some for highly compressible materials.
- It is very important to use appropriate model for a given material.
- Physically, if the material is known to be incompressible in nature, then using a material model where the deviatoric and volumetric terms are not tightly coupled is preferred (e.g., Mooney-Rivlin, Yeoh, Ogden, etc.,).
- If the material is known to be compressible, then using a material model where the deviatoric and volumetric terms are tightly coupled is preferred (e.g., Ogden-foam, Blatz-ko, etc.,).

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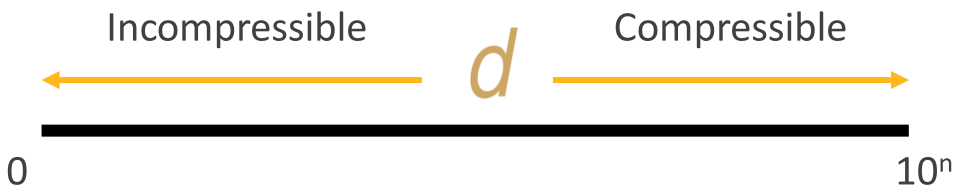
- Proper usage of volumetric behavior is crucial for accuracy of the calculations.
- It's important to understand the physical meaning of the incompressibility parameter, d, which is used in most hyperelastic models.
- It is related to the initial bulk modulus, K_0 as

$$d = \frac{2}{K_0}$$

- Unlike its linear counterpart, Poisson's ratio, d is not limited to a particular range.
- As value of d decreases, the material tends to be more incompressible.



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