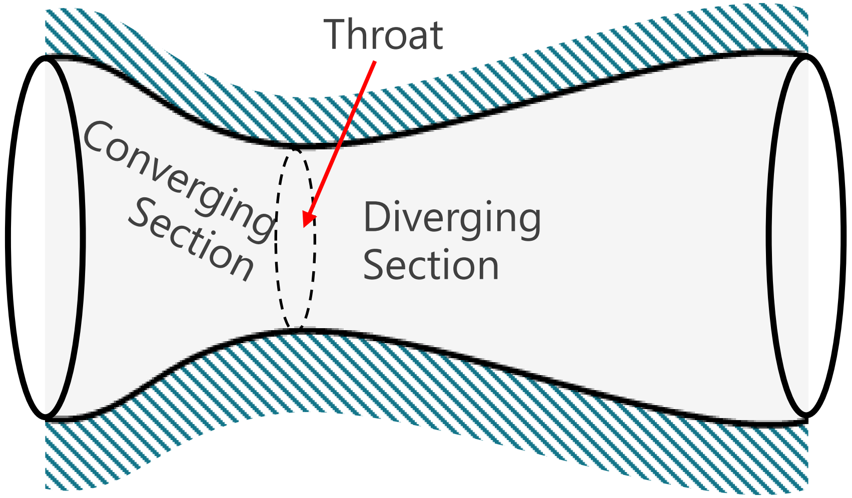
Converging-Diverging Nozzle

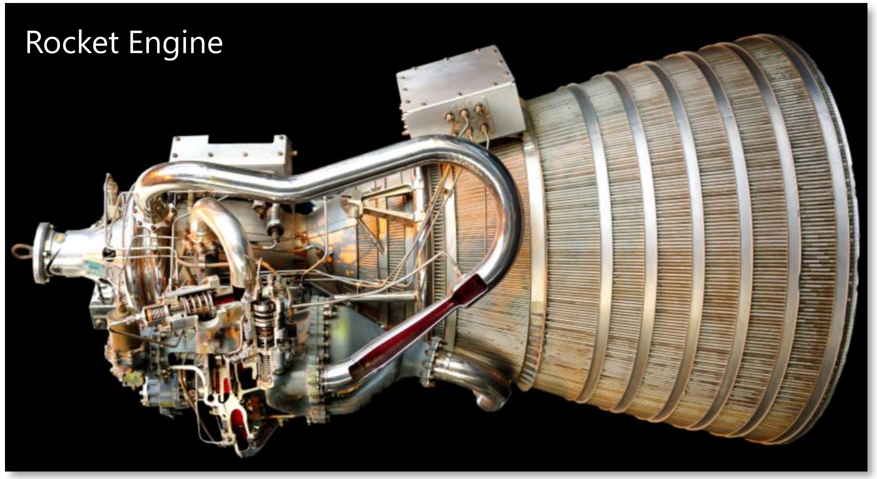
Internal Compressible Flows – Lesson 4

Ansys

Intro

- In this lesson we will analyze the flow in a converging-diverging (CD) Nozzle.
- A CD Nozzle is a variable area passage which is used to accelerate gases to higher supersonic speeds.
- It consists of a converging section with minimum area occurring at a specific location called the throat.
- Downstream of the throat, the cross-sectional area starts to increase, thus creating the diverging section of the CD nozzle.
- It is most commonly used in propulsion systems such as rocket engines and after-burners of jet engines.
- It is also used in supersonic wind tunnels.



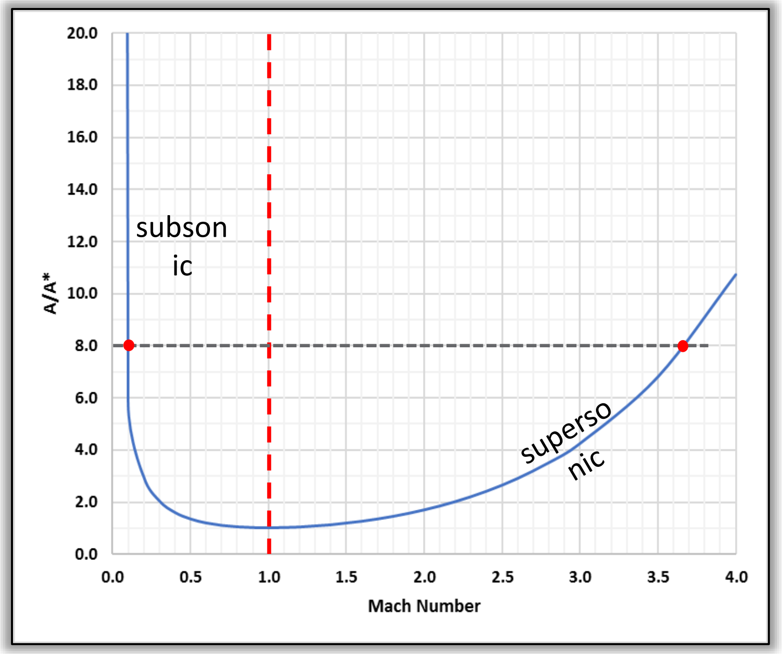


Area – Mach Number Relation

- Using the continuity and the isentropic relations, we can get the following equation relating the area to the Mach number:

$$\frac{A}{A^*} = \frac{1}{M}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

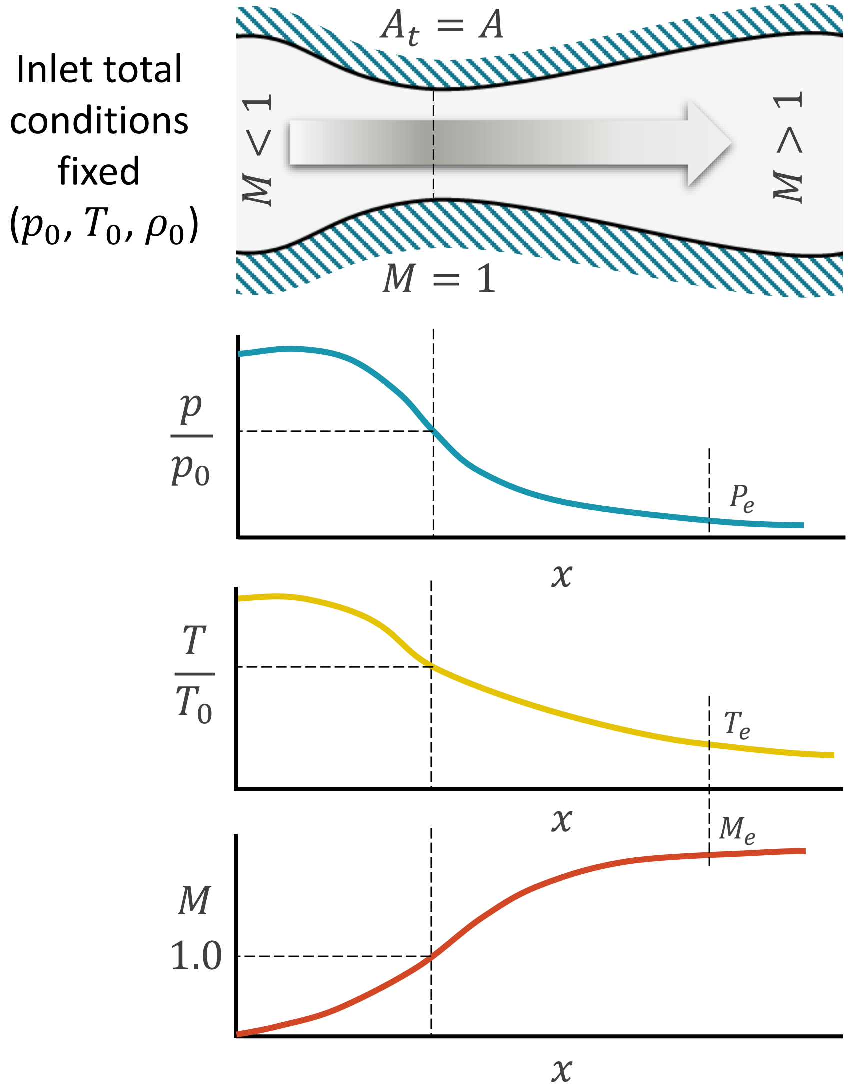
- This equation is called the area – Mach number relation, and it shows that $M = f(A/A^*)$, i.e., the Mach number at any location in the duct is a function of the ratio of the local duct area to the sonic throat area.
- Note that there are two values of M that correspond to a given area ratio (A/A^*), a subsonic and a supersonic value.
- The solution to this equation is plotted in the graph on the right, clearly showing the subsonic and the supersonic branches.



Variation of Mach Number with Area

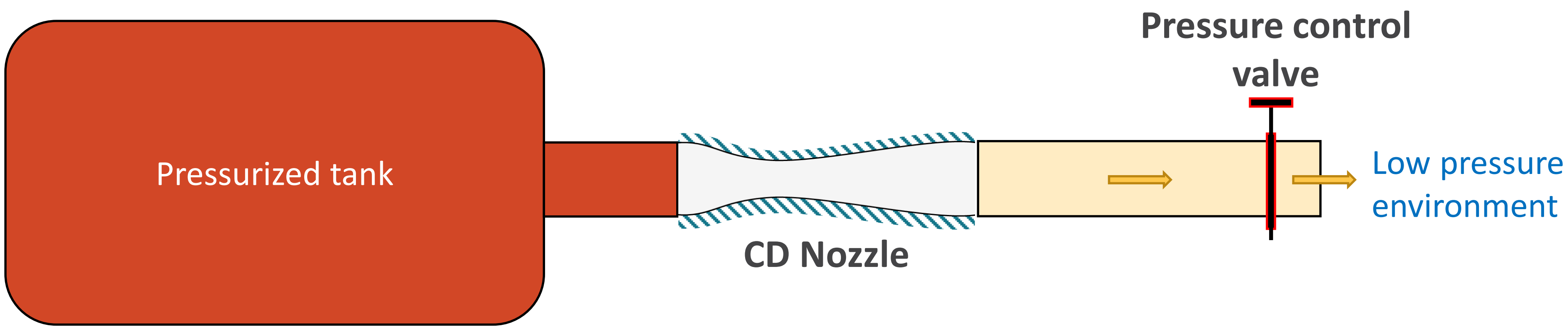
1D Flow through a CD Nozzle

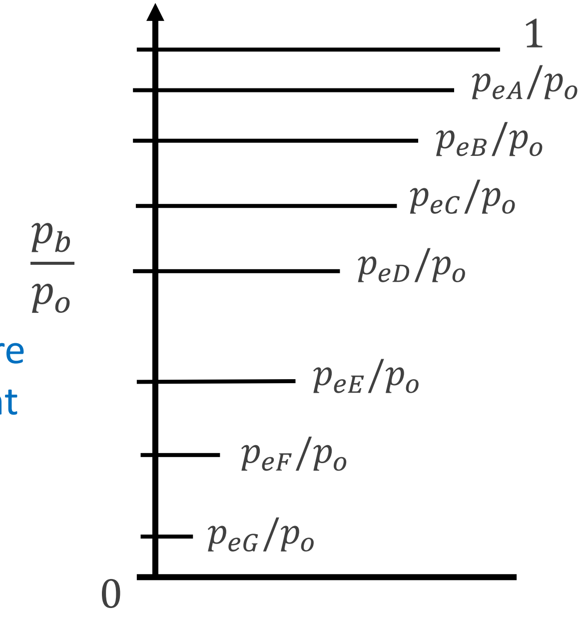
- Consider a CD nozzle such that the area ratio at the inlet $A_i/A^* \rightarrow \infty$. The inlet station feeds from a large reservoir operating at the p_0, ρ_0 and T_0 (stagnation properties).
- In the convergent portion of the nozzle, the subsonic flow is accelerated, and the Mach number is dictated by the local value of A/A^*. At the throat, where $A_t = A^*$, we get $M = 1$.
- In the divergent portion of the nozzle the flow expands supersonically, and again the Mach number (supersonic now) is governed by the local value of A/A^*.
- The resulting variations of the pressure, temperature and Mach number follow a monotonic increase or decrease as shown on the right.
- P_e, T_e and M_e are the pressure, temperature and Mach number at the nozzle exit. In an ideal situation the exit pressure P_e is equal to the ambient pressure at the exit (back pressure, p_b) – this situation is referred to as "design condition."
- Next, we will evaluate the effect of different pressure ratios.



CD Nozzle Model

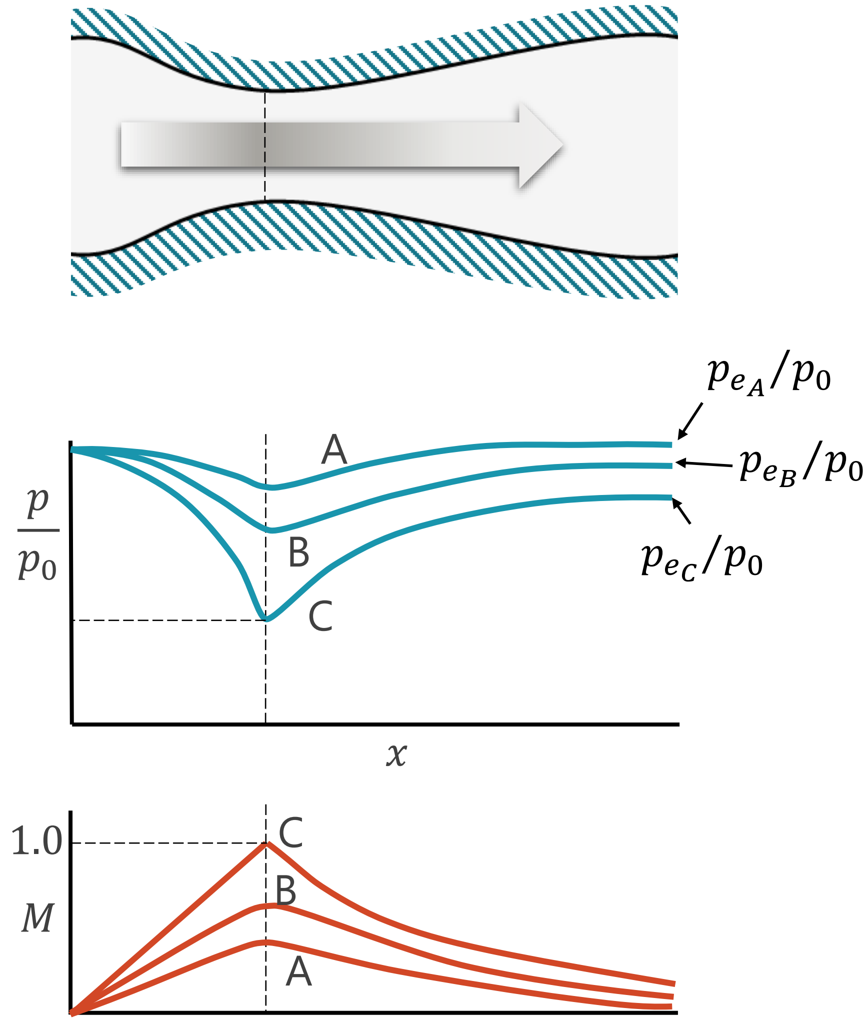
- For our analysis, we will use the physical model of the CD nozzle as shown in the figure.
- The nozzle is connected to a tank supplying the inlet station with fixed total pressure and temperature.
- A valve attached to the nozzle exit is adjusted to vary back pressure.
- Except for two special conditions which we will discuss later, the nozzle exit pressure will always be equal to the back pressure. For the purpose of our discussion, we will be using the two terms interchangeably and make the distinction when necessary.
- Let us analyze what happens to the flow as we open the valve and reduce the back pressure from $p_{eA} > p_{eB} > p_{eC} > p_{eD} > p_{eE} > p_{eF} > p_{eG}$.



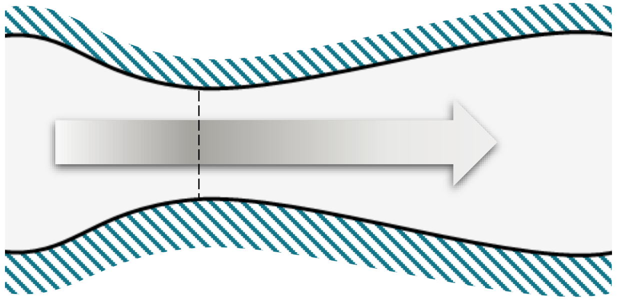


Fully Subsonic Flow

- Initially, $p_e = p_b = p_0$. Therefore, there will be no flow in the nozzle.
- When the back pressure is slightly reduced such that $p_e = p_b = p_{eA}$, a low-speed flow is established in the nozzle, as indicated by Case A.
- As we start decreasing the back/exit pressure, p_b/p_e, from p_{e_A} to p_{e_B}, this flow starts accelerating as shown by case B.
- If the flow in the entire nozzle is subsonic, then the flow at any axial station within the nozzle can be analyzed using the relations derived previously.
- As the exit pressure is further decreased, the flow is accelerated more, leading to a higher Mach number throughout the nozzle with the highest being at the throat. Consequently, the pressure is also lowest at the throat.
- At a certain value of $p_b = p_e = p_{eC}$, the flow just becomes sonic at the throat (Case C). At this point $A_t = A^*$ and $M_t = 1$.

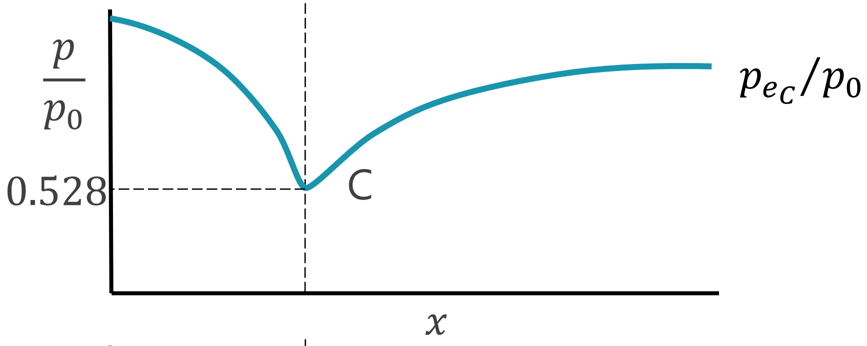


Choked Flow – Converging Section

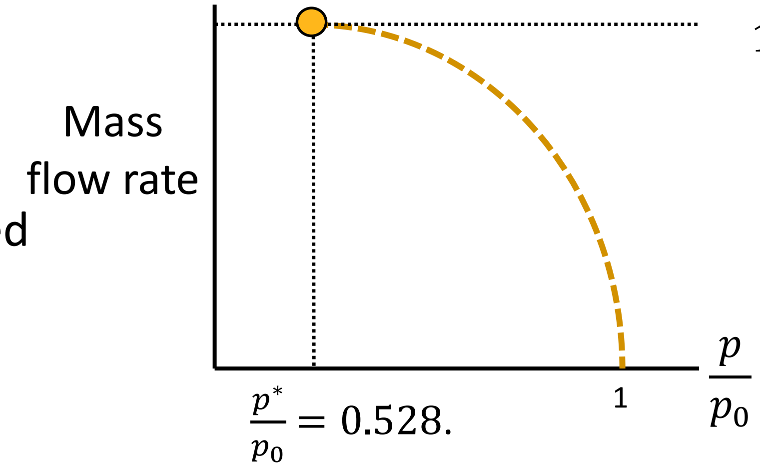


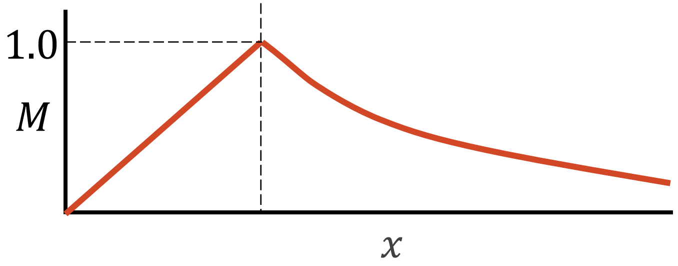
- Decreasing the exit pressure from p_{eA} to p_{eC}, the mass flow through the nozzle ($\dot{m} = \rho_t A_t V_t$) increases until sonic conditions are reached at the throat corresponding to p_{eC}.
- The mass flow rate through the CD nozzle corresponding to condition C, i.e., sonic flow attained at the throat, is given by:

$$\dot{m} = \rho_t A_t V_t = \rho^* A^* V^*$$



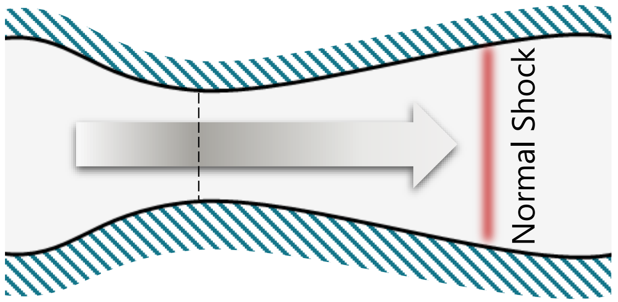
- If the back pressure is further reduced, the Mach number at the throat cannot increase beyond 1.0 and the mass flow remains constant. This condition is referred to as choked flow.
- For $\gamma = 1.4$, the sonic flow at the throat corresponds to the pressure ratio of $p^*/p_0 = 0.528$.

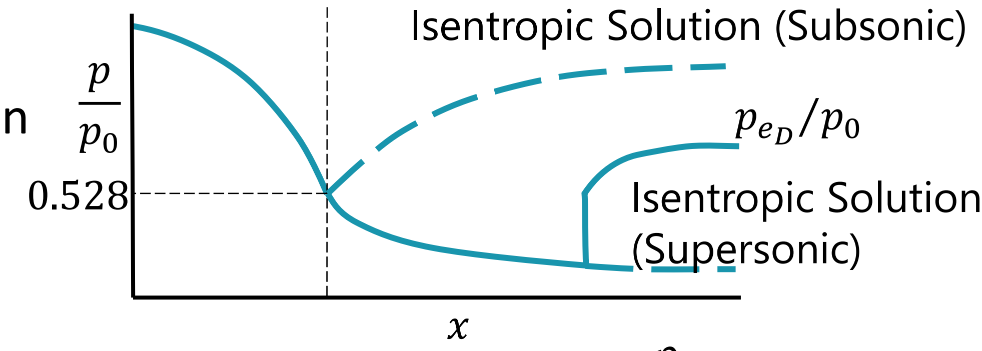


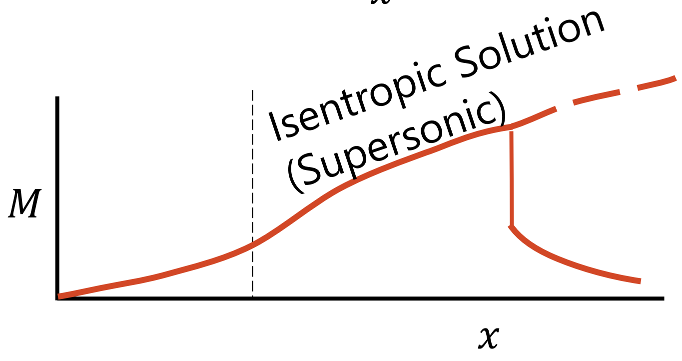


Choked Flow – Diverging Section

- If the exit pressure is further decreased to a value below the sonic condition, $p_{eD} < p_{eC}$, the flow remains unchanged in the converging section. However, an interesting flow phenomenon occurs in the diverging section of the nozzle.
- No isentropic solution is possible in the divergent duct until the nozzle exit pressure is adjusted to the specified low value as shown earlier for the design condition case.
- For values of exit pressure above the supersonic design condition value, but below p_{e_C}(sonic condition), a normal shock wave is observed inside the diverging section as shown.
- The region ahead of the shock in the diverging section is supersonic, while behind the shock the flow is subsonic. As a result, the Mach number decreases and the static pressure increases toward the exit.

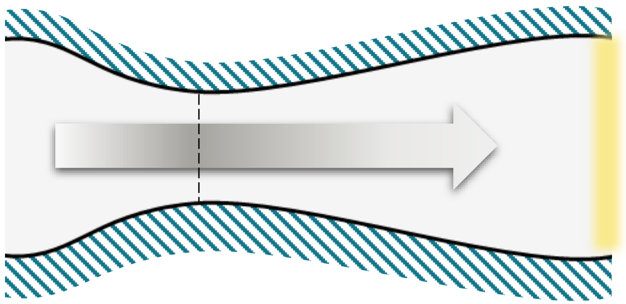


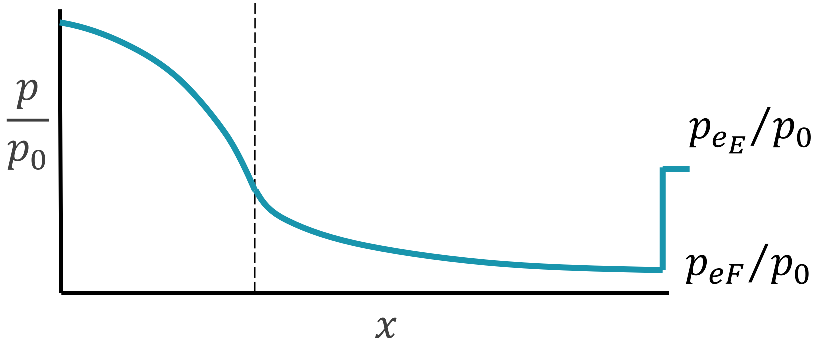


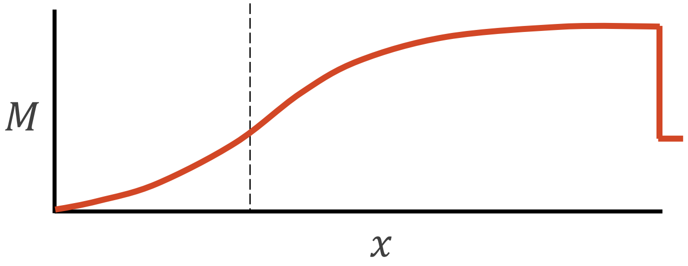


Choked Flow – Diverging Section (cont.)

- The location of the normal shock is such that it allows the exit pressure to attain the prescribed value $p_{e_D} = p_b = p_e$ at the nozzle exit.
- The flow accomplishes this via the increase of static pressure across the shock wave and the pressure increase due to the subsonic flow in the remaining diverging section.
- As we decrease the exit pressure further from p_{eD}, the shock wave moves downstream and closer to the nozzle exit.
- At a certain value $p_{e_E} < p_{eD}$ the shock will be located precisely at the exit.
- Note that in this figure p_{eF} represents the proper isentropic value for the design exit Mach number and it exists just upstream of the normal shock.
- From this condition, the back pressure and exit pressure will be distinct.
- Now let's reduce the backpressure even further and see what happens to the shock at the exit.

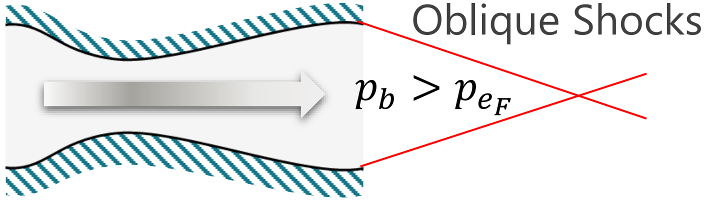




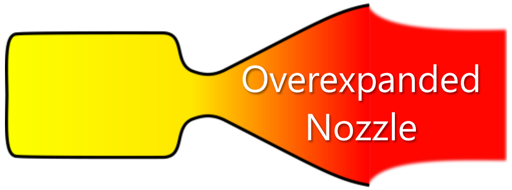


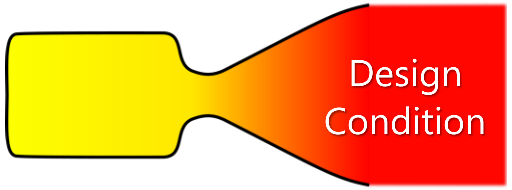
Overexpanded and Underexpanded Nozzles

- When the downstream backpressure (p_b) is further reduced such that $p_{eF} < p_b < p_{e_E}$ then the flow inside the nozzle is fully supersonic and isentropic. However, the nozzle exit pressure is maintained at p_{eF}

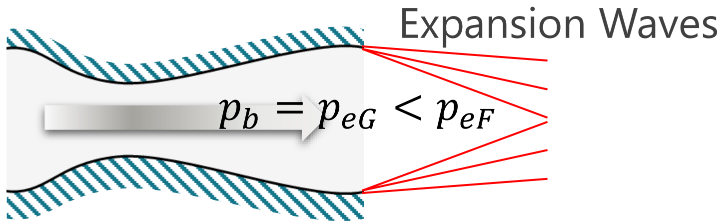


- The increase of exit pressure (p_{eF}) to the backpressure (p_b) takes place across an oblique shock attached to the nozzle exit, outside the duct itself as shown. The nozzle in this case is said to be overexpanded, because the pressure at the exit has expanded below the back pressure ($p_{eF} < p_b$).

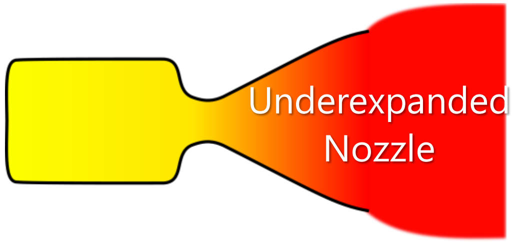




- If the back pressure (p_b) is further reduced, we reach a stage where $p_b = p_{eF} = p_e$. At this point the nozzle is said to be operating at design conditions, as we get perfect supersonic expansion.

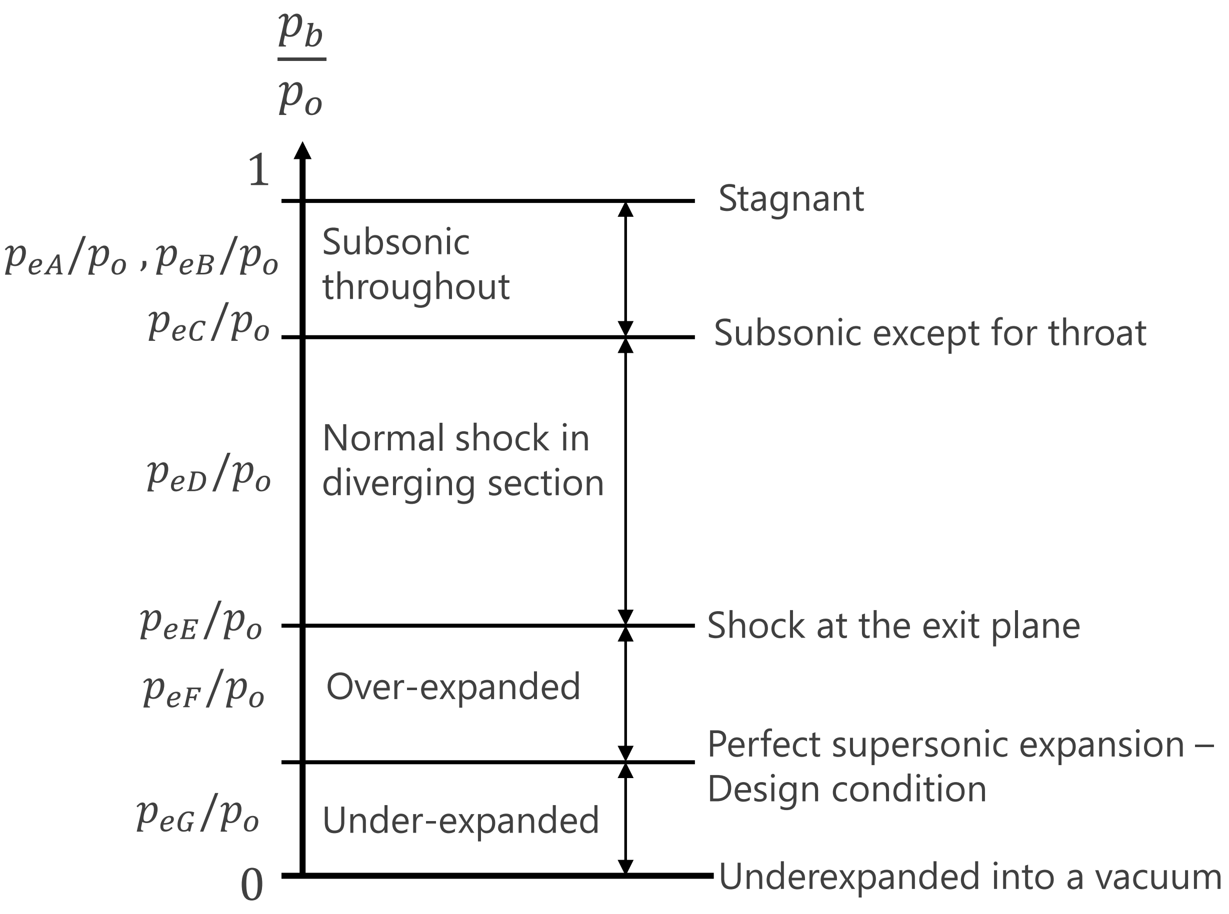


- If the back pressure is further reduced to p_{eG}, the flow adjustment takes place across expansion waves outside the duct as shown.



- In this case, the nozzle is said to be underexpanded, because the exit pressure is higher than the back pressure, ($p_{eF} > p_b$) and the flow is capable of further expansion after exiting the nozzle.

Effect of Backpressure on CD nozzle Flow



1D Flow Solution for CD Nozzle

- Now that we understand the flow physics of a CD nozzle, lets look at the overall process of analyzing the flow quantitatively.
- We begin the analysis by assuming the flow through the nozzle is entirely isentropic. Therefore, total properties are constant: $p_{01} = p_{02}, T_{01} = T_{02}, \rho_{01} = \rho_{02}$
- Using the isentropic relations and the known exit pressure p_2, we can compute M_2:

$$\frac{p_{02}}{p_2} = \left(1 + \frac{\gamma - 1}{2} M_2^2\right)^{\gamma/(\gamma-1)}$$

- We now calculate A^* from the area ratio equation, using A_2 and M_2. This is a provisional critical value which we will use to determine if in fact the flow is isentropic and fully subsonic.
- Next, we compare the actual throat area A_t with A^*. Two possibilities exist:
  - $A_t > A^*$ → This indicates that the throat is too large for sonic flow and thus the entire nozzle flow is subsonic.
  - $A_t < A^*$ → This indicates that the throat is too small for subsonic flow and thus the nozzle is choked. For this scenario, we can split the analysis into two parts: (A) the converging section and (B) the diverging section.
- Let's consider each of the above cases in detail.

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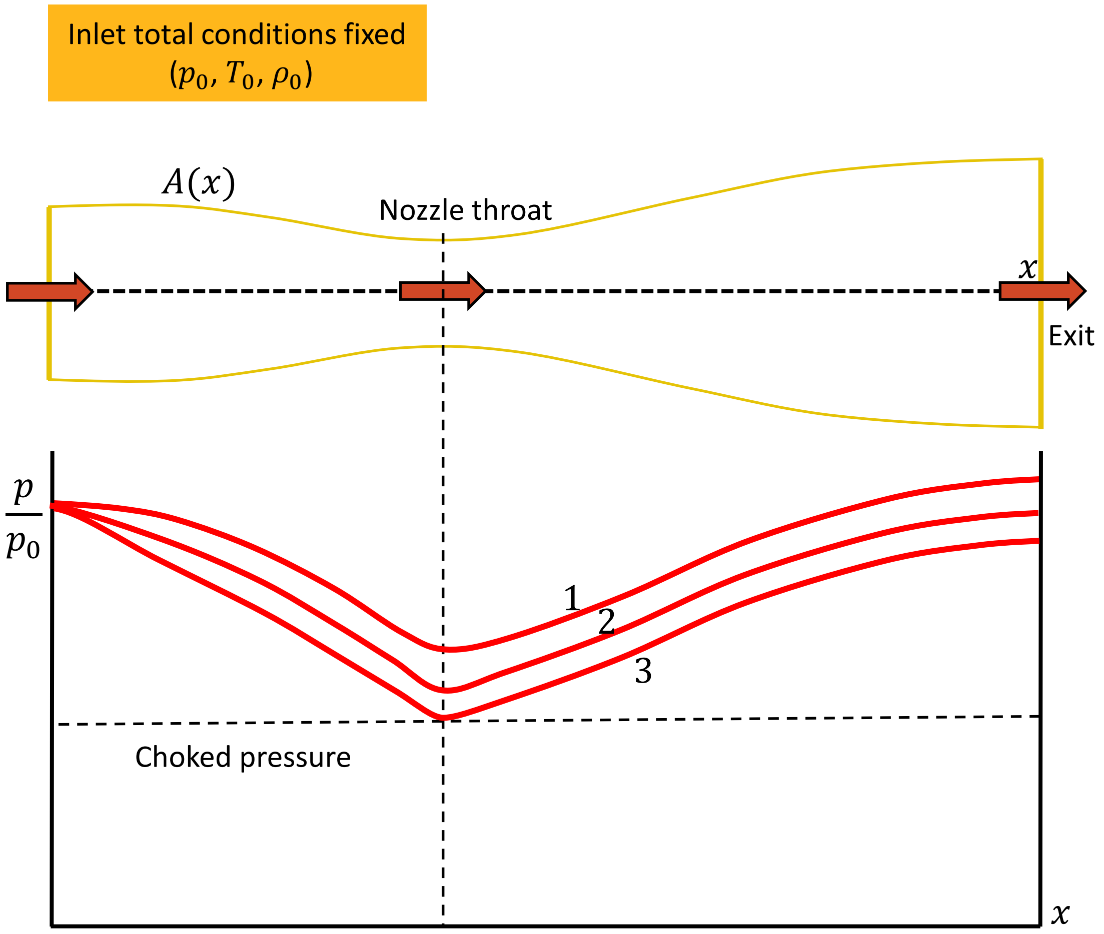
Fully Subsonic Flow

- If the flow in the entire nozzle is subsonic, then calculating the flow at any axial station within the nozzle is straightforward. Just use the computed A^* from the first step as a reference quantity.
  - For any station, compute the area ratio A/A^* and calculate the Mach number from the area ratio equation. You can use tables, charts or solve the equation numerically for M.

Note that you only consider the subsonic solution here.

  - Knowing the total properties and Mach number, calculate all flow properties and the mass flow rate. As a check, make sure your calculated mass flow rate is consistent at each station.

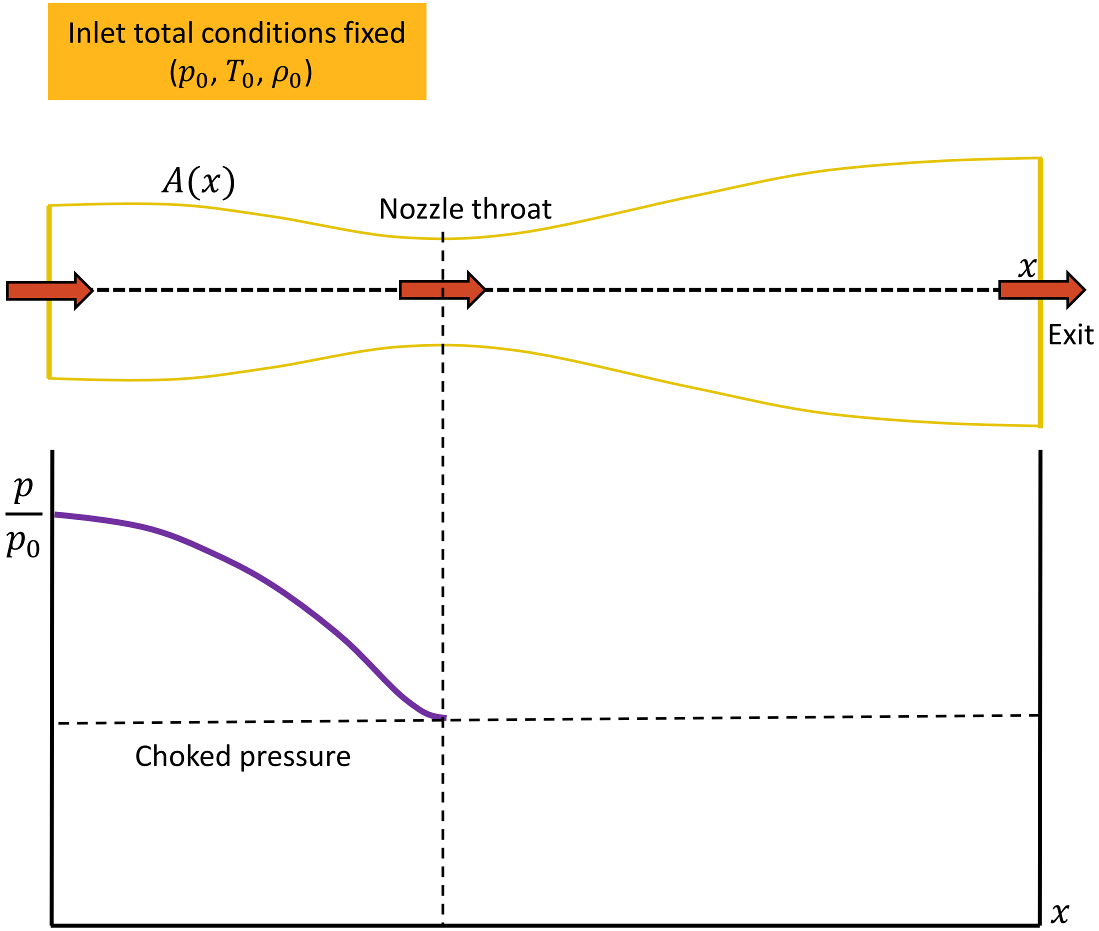


Choked Flow – Converging Section

- If the throat is choked, then the physical area A_t and the reference area A^* are the same.

$$A^* = A_t$$

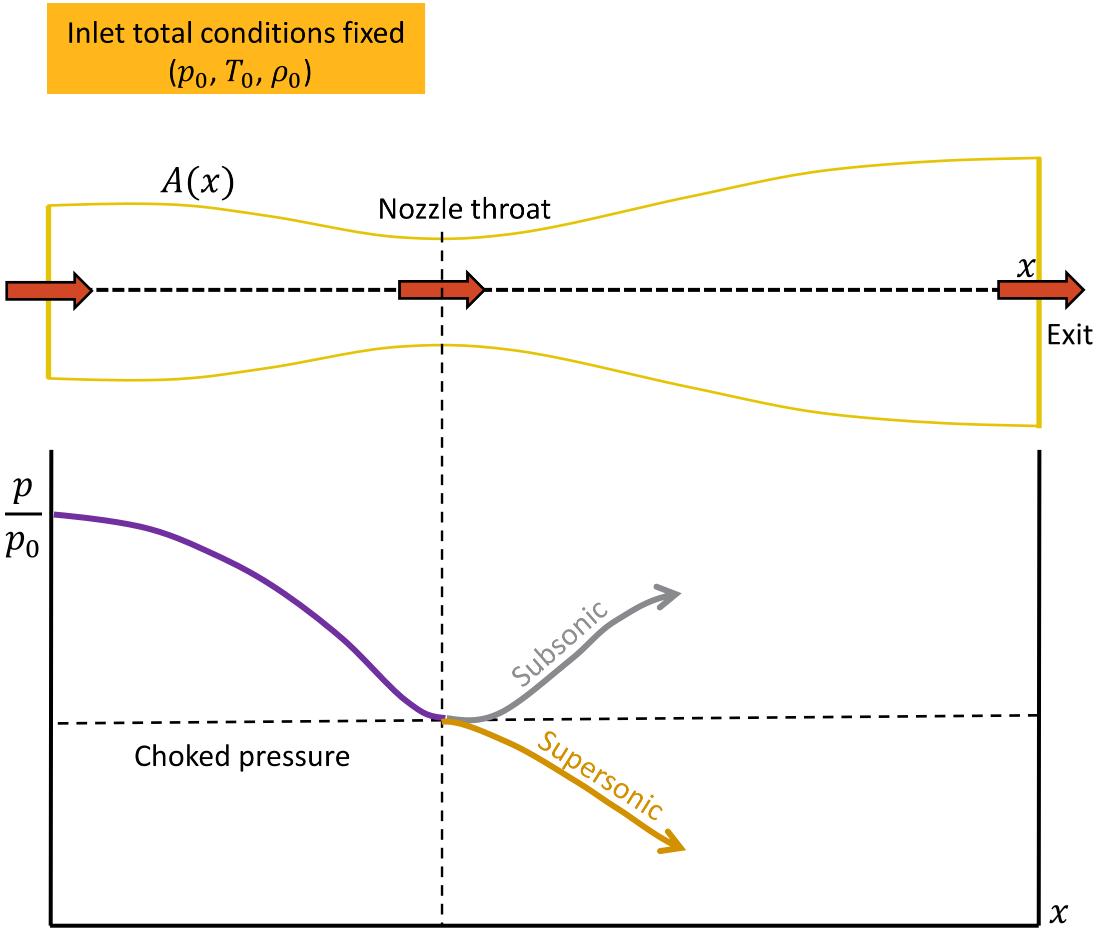
- Note that the flow in the converging section is subsonic and isentropic. Hence, we can calculate any station from the inlet to the throat as defined previously for the fully subsonic case.
  - For any station, compute the area ratio A/A^* and calculate the Mach number from the area ratio equation. Use the subsonic solution.
  - Knowing the total properties and Mach number, calculate all flow properties and the mass flow rate. Make sure your mass flow rates are consistent with the inlet as a check.



Choked Flow – Diverging Section

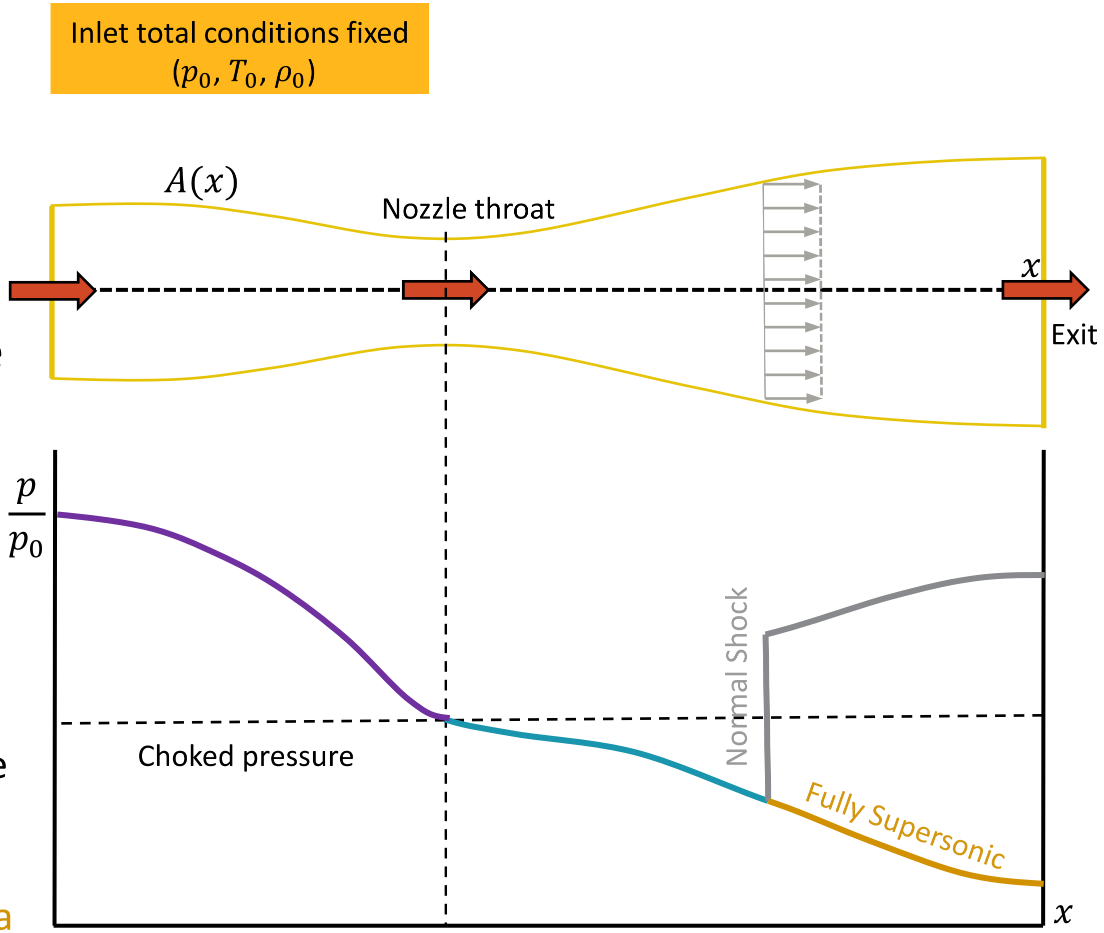
- In the diverging section of the nozzle, there are two possibilities:
  - The flow at the exit is supersonic.
  - The flow at the exit is subsonic.
- As discussed previously, in the subsonic case, there must be a normal shock wave in the diverging section as the flow is supersonic when it enters the diverging section.
- The subsonic flow downstream of the shock will then diffuse (slow down).

Note that which case will occur depends entirely on the prescribed exit pressure!



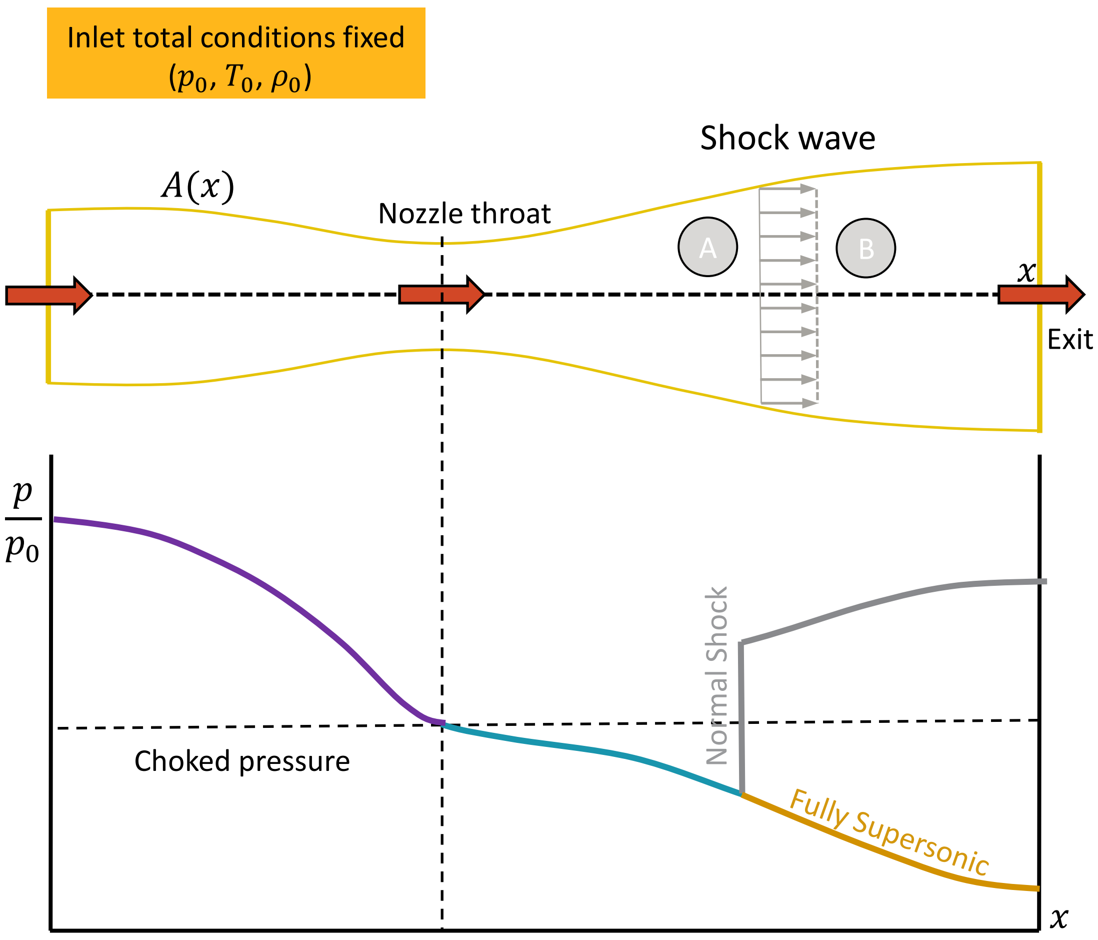
Choked Flow – Diverging Section (cont.)

- Let's look at the case of supersonic flow at the exit.
  - We first compute a provisional exit supersonic Mach number knowing the exit area ratio A_2/A^*. We will denote this as M_2'. Note that we want the supersonic solution to the area ratio equation.
  - From this provisional Mach number, we calculate the theoretical exit pressure p_2' using the isentropic relation for pressure ratio (since no shocks exist in the diverging passage and $p_{01} = p_{02}$).
  - Then compare the computed p_2' with our prescribed p_2:
    - $\boldsymbol{p_2 < p_2'}$ - For this case, the flow expands outside of the nozzle, and thus the actual pressure that will exist at the nozzle exit plane is p_2'. Hence, our previous analysis for the supersonic exit condition is used.
    - $p_2 > p_2'$ - For this case, it is clear that the flow must undergo a normal shock wave somewhere in the diverging passage and then diffuse subsonically to the prescribed exit pressure. We will discuss this case next.



Normal Shock in the Diverging Section

- The problem in this case is that we don't know the exact location of the normal shock wave. However, we can compute the position using a trial and error process in conjunction with the shock wave relations, shown below for reference. For clarity, we will denote each side of the shock by the letters A (upstream) and B (downstream).



$$\frac{p_A}{p_B} = \frac{2}{\gamma+1}M_A^2 - \frac{\gamma-1}{\gamma+1}$$

$$\frac{\rho_B}{\rho_A} = \frac{(\gamma+1)M_A^2}{2+(\gamma-1)M_A^2}$$

$$\frac{T_B}{T_A} = \left(\frac{2\gamma}{\gamma+1}M_A^2 - \frac{\gamma-1}{\gamma+1}\right)\left[\frac{\gamma-1}{\gamma+1} + \frac{2}{(\gamma+1)M_A^2}\right]$$

$$M_B = \sqrt{\frac{M_A^2 + \frac{2}{\gamma-1}}{\frac{2\gamma}{\gamma-1}M_A^2 - 1}}$$

Trial and Error Method for Finding The Shock Wave Position

Begin by guessing an axial position x in the diverging passage. A shock wave is fixed at this position with upstream downstream stations A and B.

Since the flow from the nozzle inlet to point A is isentropic, we know the total properties at A. From the area ratio equation we can calculate M_A using the supersonic solution.

Using M_A and the total properties at A we then calculate the static properties at A (p_A, T_A, ρ_A).

Now apply the shock wave equations to obtain p_B, T_B, ρ_B, and M_B knowing M_A and the static properties. We can obtain the total quantities p_{0B}, T_{0B}, ρ_{0B} which will prevail through the remainder of the nozzle (which flows isentropically).

From the area $A(x)$ and M_B we now calculate a new A_B^*. This reference critical area will also prevail from the downstream side of the shock to the exit.

Using the exit area A_2, we calculate A_2/A_B^* and then obtain a provisional subsonic M_2, again using the area ratio equation.

Finally, using the isentropic relation for total to static pressure ratio, we solve for a provisional exit pressure p_2'.

- If p_2' is greater than the prescribed exit pressure p_2, the shock must be closer to the throat and the assumed axial position x is reduced;
- Otherwise, we increase x. A small increment Δx is chosen for these adjustments.

Update x

Summary

- In this lesson, we analyzed 1D compressible flow in a converging-diverging (CD) nozzle.
- Several solutions are possible depending on the boundary conditions (particularly the exit static pressure).
- The solution methods can easily be codified in a computer program, spreadsheet or similar tool.
- Despite the widespread use of numerical methods for flow solutions, these methods are useful for quick design calculations as well as illustrating key features of the CD nozzle flow field.

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