Turbulent Boundary Layers

Basics of Turbulent Flows – Lesson 6

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Intro to Turbulent Boundary Layers

- The turbulent boundary layer problem does not have an analytical solution.
- Thus our analysis will be based on integral methods describing fundamental conservations laws, physics-based arguments, method of matching solutions, and empirical correlations.

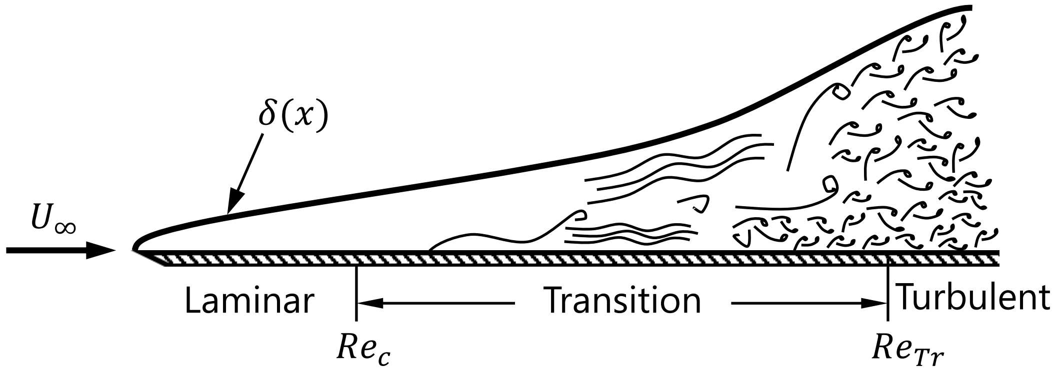


Illustration of laminar to turbulent transition on a flat plate

2D Turbulent Boundary Layer Equations

- For a fully turbulent, wall-bounded flow, we will employ the 2D, dimensional form of the incompressible Reynolds-Averaged Navier-Stokes equations in Cartesian coordinates.
- We will perform an order of magnitude analysis to eliminate negligible terms.

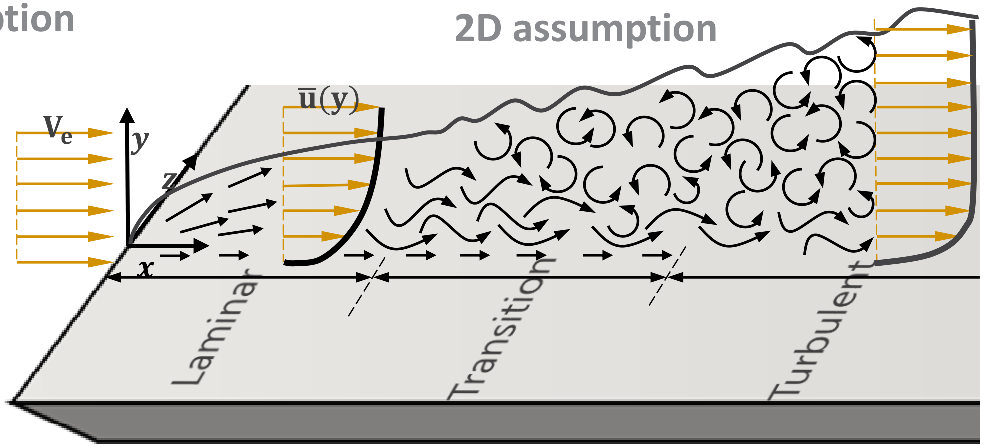
$\delta(x) \ll x$ $\qquad$ $\frac{\partial}{\partial x} \ll \frac{\partial}{\partial y}$

Order of magnitude assumption

$\overline{w} = 0$ $\qquad$ $\frac{\partial \overline{w'^2}}{\partial z} = 0$

2D assumption

Note that $\overline{w'^2}$ is, strictly speaking not zero, but its z-derivative is.



2D Turbulent Boundary Layer Equations

- The RANS equations then reduce to the following boundary-layer approximation:

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0$$

$$\rho\left(\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y}\right) = -\frac{\partial \bar{p}}{\partial x} + \frac{\partial}{\partial y}\left(\mu\frac{\partial \bar{u}}{\partial y} - \rho\overline{u'v'}\right)$$

$$\frac{\partial \bar{p}}{\partial y} = -\frac{\partial \overline{v'^2}}{\partial y}$$

Reynold's stress term

This term is assumed to be small across the boundary layer (~0.4% of freestream dynamic pressure)

- Assuming the inviscid region outside the boundary layer where the Bernoulli's equation is satisfied:

$$dp_e \approx -\rho V_e dV_e$$

and the momentum boundary layer equation becomes:

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} = V_e\frac{dV_e}{dx} + \frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial \bar{u}}{\partial y} - \rho\overline{u'v'}\right)$$

Boundary conditions:

$\bar{u}(x,0) = \bar{v}(x,0) = 0$ *no-slip*

$\bar{u}(x,\delta) = V_e(x)$ *meanflow matching*

Turbulent Boundary Layer Integral Relationships

- The expressions for displacement and momentum thicknesses derived using integral forms of conservation of mass and momentum equations* are:

$$\delta^* = \int_0^\infty \left(1 - \frac{\bar{u}}{V_e}\right) dy \qquad \theta = \int_0^\infty \frac{\bar{u}}{V_e}\left(1 - \frac{\bar{u}}{V_e}\right) dy$$

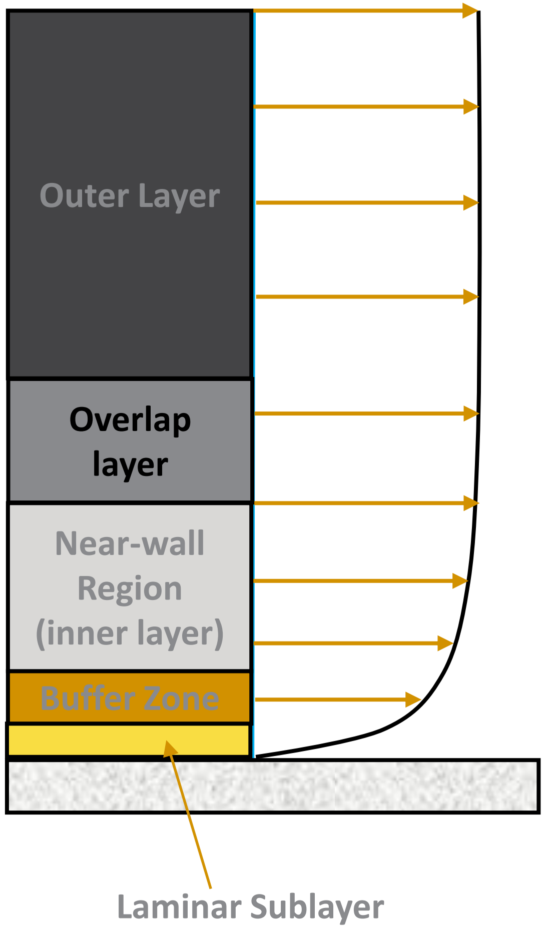
$$H = \frac{\delta^*}{\theta} \qquad \frac{d\theta}{dx} + (2 + H)\frac{\theta}{V_e}\frac{dV_e}{dx} = \frac{\tau_w}{\rho V_e^2} = \frac{C_f}{2}$$

- The integral equation for the momentum thickness contains unknown fluctuations, which can be obtained by using turbulence models, discussed in the next lesson.

*See appendix at the end of the presentation for more details

Turbulent Velocity Distribution

- Prandtl's mixing length hypothesis can be used to deduce the behavior of the turbulent velocity profile. The observed behavior of the profile shows that it can be divided into three basic zones:
- The region that is influenced by the presence of the wall is called the Inner Layer or the Near-wall region. It should be noted that the top of the Near-Wall region only comprises about 15% of the total turbulent boundary thickness! This zone is further divided into two categories
  - It is known that very close to a wall the boundary layer is essentially laminar. This thin zone is known as the Laminar Sublayer.
  - At the top of the laminar sublayer, the flow begins to feel the effect of turbulence, though laminar influence is still present. This zone is called the Buffer Zone.
  - The freestream turbulence effects slowly increase in strength through the inner layer, and the laminar behavior is lost, although the presence of the wall still influences the flow.
- Going beyond the Near-wall Region leads us to the edge of the boundary layer. There, freestream turbulence effects dominate, and this region is thus called the Outer Layer.
- The inner layer is transitioned to the outer layer through the Overlap Layer.



Turbulent Boundary Layer: Inner, Outer and Overlap Layers

- Prandtl and Karman reasoned the following:
  - The inner layer depends on wall shear stress, density and viscosity, and distance from the wall, but not on meanflow:

$$\frac{\bar{u}}{v^*} = f_{inner}\left(\frac{yv^*}{\nu}\right), \qquad v^* = \left(\frac{\tau_w}{\rho}\right)^{1/2}$$

Here v^* - wall-friction velocity, and it is frequently used in turbulent flow analysis.

  - The outer layer depends on the mean flow pressure gradient, layer thickness and wall shear stress, but is independent of viscosity:

$$\frac{V_e - \bar{u}}{v^*} = f_{outer}\left(\frac{y}{\delta}, \zeta\right), \qquad \zeta = \frac{\delta}{\tau_w}\frac{dp_e}{dx}$$

  - The overlap layer is a smooth blend of inner and outer layers over a finite distance:

$$\bar{u}_{inner} = \bar{u}_{outer}$$

Turbulent Boundary Layer: Inner, Outer and Overlap Layers (cont.)

- Matching inner and outer solutions lead to the appearance of logarithmic functions, and the expressions for the inner and outer layers become:

$$u^+ = \frac{1}{\kappa}\ln y^+ + B, \qquad V_e^+ - u^+ = -\frac{1}{\kappa}\ln\frac{y}{\delta} + A(\zeta)$$

Inner layer — Outer layer

$$\kappa \approx 0.4, \; B \approx 5.5$$

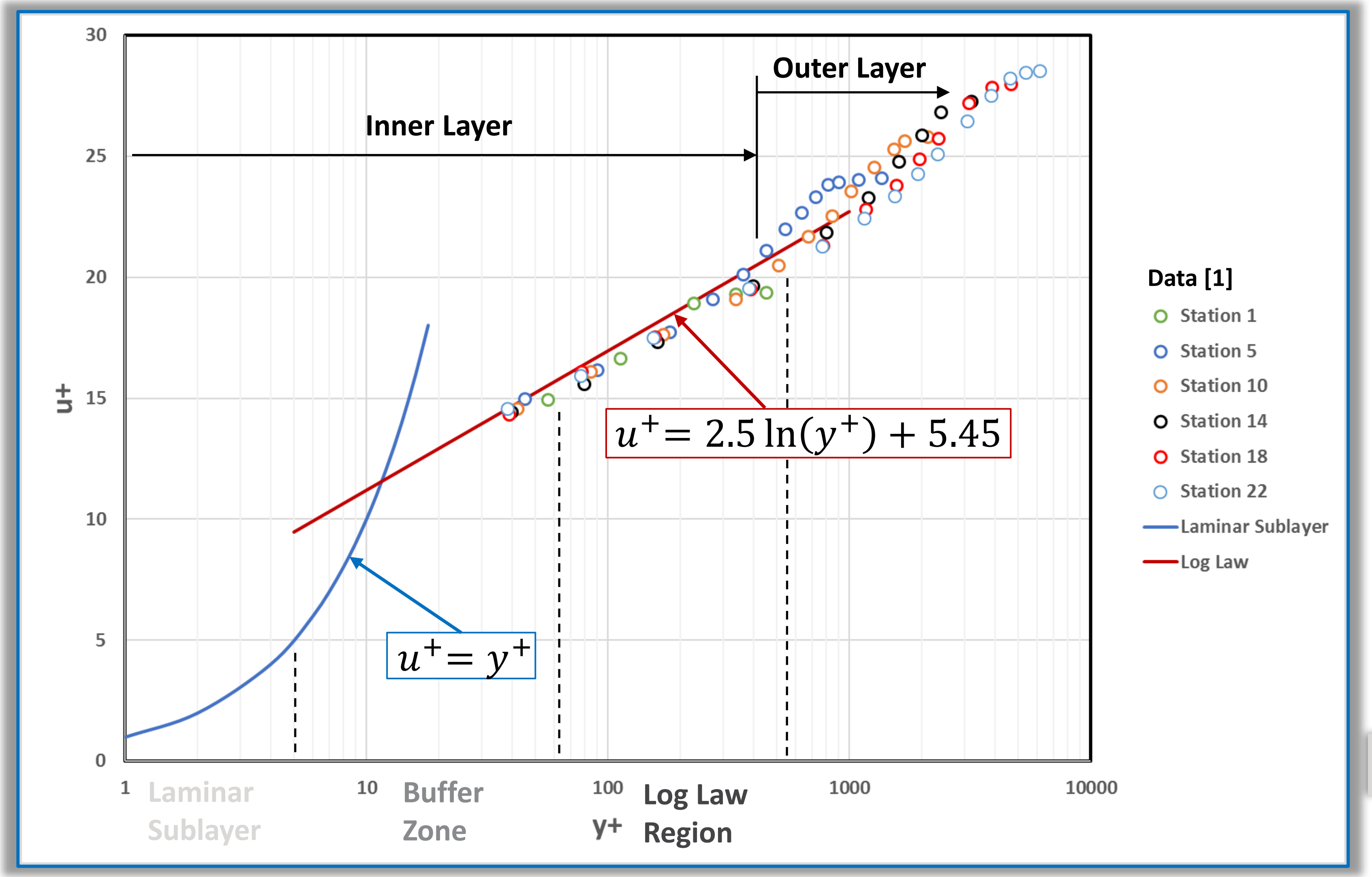
Empirical constants

$$u^+ = \frac{\bar{u}}{v^*}, \qquad y^+ = \frac{yv^*}{\nu}$$

Non-dimensionalization

- A in the outer layer expression is a function of the mean flow pressure gradient and potentially other parameters.
- Classical values of constants κ and B, measured back in 1930 by Prandtl's student Nikuradse, are shown.
- More recent measurements suggest slightly different values: $\kappa \approx 0.41, \; B \approx 5.0$.
- A very thin region next to the wall is dominated by the viscosity. It is called the viscous sublayer, where the velocity profile is linear: $u^+ = y^+, \qquad y^+ \leq 5$

Universal Law of the Wall



- There is a remarkable agreement between log profile distribution deduced from physics principals and experimental data!
- Extents of different layer regions are approximately:

Laminar sublayer

$0 \leq y^+ \leq 5$ | $0\% \leq y/\delta \leq 0.2\%$

Buffer layer

$5 \leq y^+ \leq 30$ | $0.2\% \leq y/\delta \leq 2\%$

Log layer

$30 \leq y^+ \leq 350$ | $2\% \leq y/\delta \leq 20\%$

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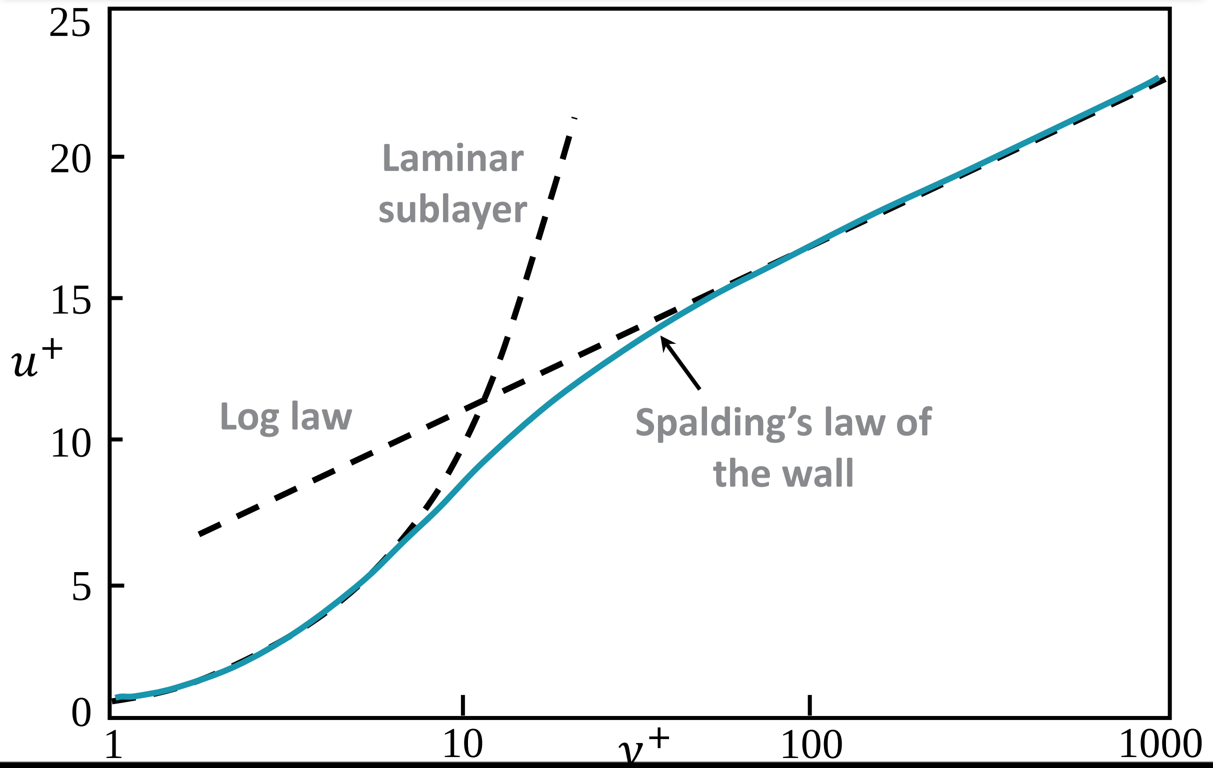
Spalding's Law of the Wall

- Spalding (1961) proposed the following composite blend of the law of the wall which is valid for the entire wall region $0 \leq y^+ \leq 350$:

$$y^+ = u^+ + e^{-\kappa B}\left[e^{\kappa u^+} - 1 - \kappa u^+ - \frac{(\kappa u^+)^2}{2} - \frac{(\kappa u^+)^3}{6}\right]$$



Outer Layer Correlation

- The outer layer is sensitive to meanflow pressure gradient and starts deviating from the log law at $y^+ \sim 350$.
- Coles (1956) noted this deviation has a wake shape, and suggested to add a wake function correction:

$$u^+ = \underbrace{\frac{1}{\kappa}\ln y^+ + B}_{\text{Log layer}} + \frac{2\Pi}{\kappa}\underbrace{f\left(\frac{y}{\delta}\right)}_{\text{Wake function}}$$

$$f(0) = 0,\ f(\delta) = 1,\ \boxed{\Pi = \kappa A/2}$$ ← Coles wake parameter

$$0.42\Pi^2 + 0.76\Pi - 0.4 = \frac{\delta^*}{\tau_w}\frac{dp_e}{dx}$$ ← Correlation for Π

- A commonly used curve fit for the wake function is:

$$f\left(\frac{y}{\delta}\right) \approx \sin^2\left(\frac{\pi}{2}\frac{y}{\delta}\right) \approx \boxed{3\left(\frac{y}{\delta}\right)^2 - 2\left(\frac{y}{\delta}\right)^3}$$ ← Taylor series approximation convenient for integration

- Integration of this velocity profile over the entire boundary layer gives estimates for δ^*, θ and C_f :

$$\frac{\delta^*}{\delta} = \frac{1+\Pi}{\kappa\lambda}$$

$$\frac{\theta}{\delta} = \frac{\delta^*}{\delta} - \frac{2 + 3.2\Pi + 1.5\Pi^2}{\kappa^2\lambda^2}$$

$$\lambda = \left(\frac{2}{C_f}\right)^{1/2} = \frac{1}{\kappa}\ln\left(\frac{Re_\delta}{\lambda}\right) + B + \frac{2\Pi}{\kappa}$$

Turbulent Boundary Layer Correlations

- Assuming meanflow equilibrium (zero pressure gradient), the velocity profile can be described by the wake law with $\Pi = 0.45$, and profile evaluation at $y = \delta$ gives the relation between C_f and Re_δ:

$$\left(\frac{2}{C_f}\right)^{1/2} = 2.44 \ln\left[Re_\delta \left(\frac{C_f}{2}\right)^{1/2}\right] + 7.2$$

$$C_f = 2\frac{d\theta}{dx}$$

- Further curve fitting yields the correlation for skin friction: $C_f = 0.02{Re_\delta}^{-1/6}$

- From empirical data, the one-seventh power law profile can be assumed (Prandtl):

$$\frac{\bar{u}}{V_e} = \left(\frac{y}{\delta}\right)^{1/7} \Rightarrow \frac{\theta}{\delta} = \frac{7}{72}$$

- Which, after some light math, gives the following turbulent boundary layer correlations:

$$Re_\delta = 0.16{Re_x}^{6/7} \qquad \delta/x = 0.16{Re_x}^{-1/7} \qquad C_f = 0.027{Re_x}^{-1/7}$$

- Note that original Prandtl formulas frequently quoted in the literature are less accurate!

$$\delta/x = 0.37{Re_x}^{-1/5} \qquad C_f = 0.058{Re_x}^{-1/5}$$

original Prandtl formulas (1927)

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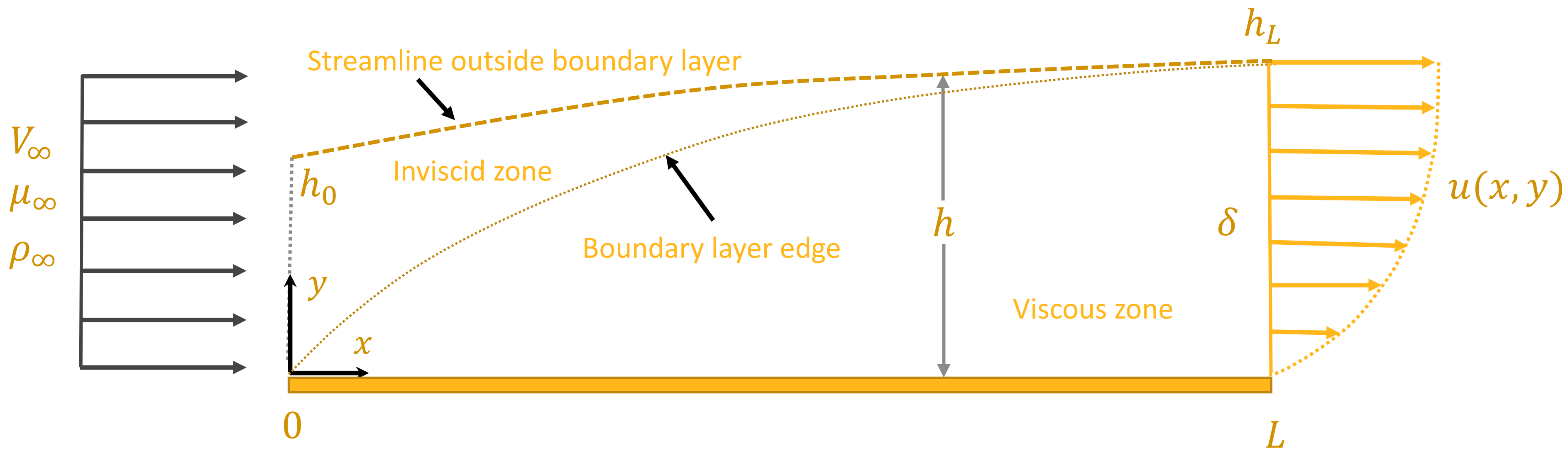
Summary

- We looked into the fundamentals of turbulent boundary layers and derived some useful estimates and correlations.
- One of the most significant topics covered was the universal law of the wall which is the backbone of numerical wall treatments in all CFD codes.
- Comprehensive discussion of turbulent boundary layers is extensive and scattered among different textbooks and journal papers, and we limited our discussion only to common derivations and correlations.

Appendix

Boundary Layer Integral Equations

- Consider the boundary layer as shown below. We define a control volume consisting of the plate, an inlet at the leading edge, a station a distance L downstream of the leading edge, and a streamline at a distance h from the plate that meets the boundary layer at station L.
- Note that the velocity profile is an input to this analysis, and so will apply to both laminar and turbulent boundary layers.



Mass Conservation Integral Equation

- Conservation of mass for the boundary layer control volume under the assumption of constant density gives:

$$\iint \vec{V} \cdot d\vec{A} = \int_0^{h_L} u\,dy - \int_0^{h_0} V_\infty dy = 0$$

- Denoting $h_L = \delta^* + h_0$, the above equation can be rewritten as:

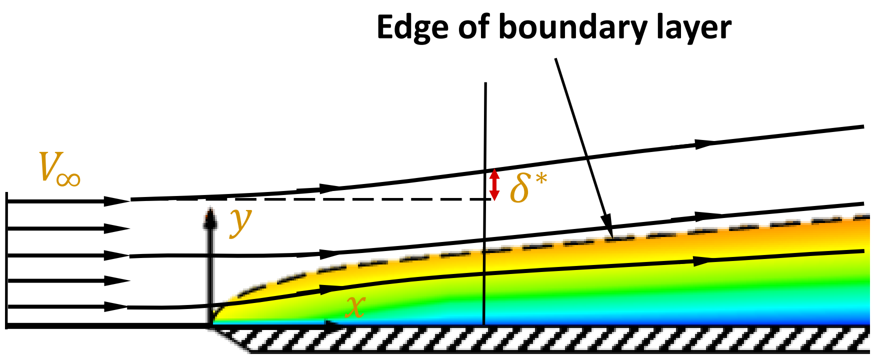
$$\delta^* = \int_0^{h_L} \left(1 - \frac{u}{V_\infty}\right) dy$$

- In the limit $h_L \to \infty$, the above equation gives the formal definition of displacement thickness:

$$\delta^* = \int_0^{\infty} \left(1 - \frac{u}{V_\infty}\right) dy$$

Boundary Layer Displacement Thickness

- The displacement thickness represents the distance by which streamlines outside of the boundary layer are displaced by the layer.
- It is a measure of blockage of the external flow due to the formation of boundary layers.
- The same definition of the displacement thickness is used for turbulent boundary layers, where the velocity is taken in time-averaged sense.
- Unlike the boundary layer thickness based on $0.99V_\infty$, which is not trivial to measure experimentally, the displacement thickness can be easily deduced from velocity measurements taken across the boundary layer.



Momentum Conservation Integral Equation

- Conservation of momentum for the boundary layer control volume under the assumption of constant density gives:

$$\iint u(\rho\vec{V}\cdot d\vec{A}) = \int_0^{h_L} u(\rho u)\, dy - \int_0^{h_0} V_\infty(\rho V_\infty) dy = -D, \qquad \text{where } D \text{ is drag}$$

- Assuming:

$$h_0 = \int_0^{h_L} \frac{u}{V_\infty} dy$$

- The above equation can be written in the limit $h_L \to \infty$ as:

$$\frac{D}{\rho V_\infty^2} = \int_0^{\infty} \frac{u}{V_\infty}\left(1 - \frac{u}{V_\infty}\right) dy$$

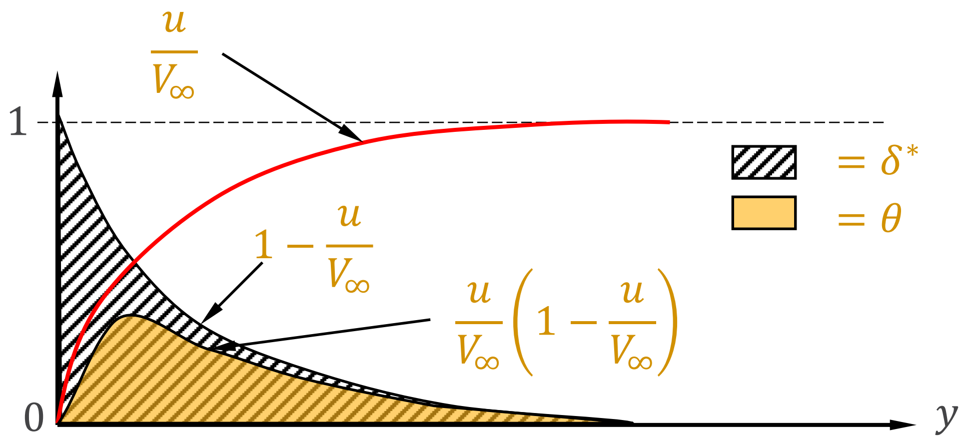
- This non-dimensional quantity is called momentum thickness of the boundary layer, which is commonly denoted by θ:

$$\theta = \int_0^{\infty} \frac{u}{V_\infty}\left(1 - \frac{u}{V_\infty}\right) dy$$

Momentum Thickness

- The momentum thickness describes the loss of momentum due to the presence of the boundary layer as compared to an equivalent inviscid flow.
- The definition of momentum thickness holds for any incompressible boundary layer, laminar or turbulent.
- For the boundary layer over a flat plate, the momentum thickness is equivalent to the non-dimensional drag.
- For an arbitrary boundary layer, however, the momentum thickness is *not* equal to the non-dimensional drag.
- The ratio of displacement thickness and momentum thickness is called the shape factor, and it is often used in the boundary layer analysis:

$$H = \frac{\delta^*}{\theta} > 1$$



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