Closure of RANS Equations

Basics of Turbulent Flows – Lesson 5

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RANS Closure Approaches

- Derivation of RANS equations posed us the problem of their closure, as nine new Reynolds stress terms also need to be modeled. There are two general approaches to close the RANS system:

1. The Boussinesq Hypothesis
   - In 1877, Joseph V. Boussinesq introduced the concept of relating the Reynolds stresses to the mean flow velocity gradients via the eddy viscosity concept.
   - Here, the Reynolds stresses are equated to laminar-like stress terms, where instead of employing a fluid viscosity we use an eddy viscosity (denoted μ_T).
   - This assumes that turbulence is isotropic (among other things), which leads to limitations on its accuracy for complex flows.
2. Reynolds Stress Transport Equation Modeling
   - A more general (and hopefully more accurate) approach is to write individual transport equations for the Reynolds stresses themselves.
   - Developing these equations (which involves modeling complex terms in the resulting Reynolds stress transport equations) is known as Reynolds Stress Modeling (RSM).

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The Boussinesq Hypothesis

- The Boussinesq hypothesis relates the Reynolds stress to mean flow velocity gradients through an eddy (or turbulent) viscosity, denoted μ_T. For incompressible flow, the Reynolds stresses can be written:

$$R_{ij} = -\rho\overline{u_i' u_j'} = \mu_T\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right) - \frac{2}{3}\delta_{ij}k$$

$$k = \frac{1}{2}\overline{u_i{}' u_i{}'}$$

← Turbulent kinetic energy

- The eddy viscosity is a function of space and time. It is not a fluid property.
- μ_T has the same dimensions as the dynamic viscosity μ.
- The Boussinesq hypothesis reduced the number of unknows from nine to two: μ_T and k.
- To make further progress, we must establish suitable hypotheses which, with the aid of empirical data, will allow us to develop models that provide μ_T and k.
- Oftentimes, the turbulent kinetic energy term is neglected in R_{ij} following the argument that shear turbulent stresses dominate over normal stresses.

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RANS Equations

- Incompressible RANS equations can now be written as:

$$\rho\left[\frac{\partial\vec{\bar{V}}}{\partial t}+\vec{\bar{V}}\cdot\nabla\vec{\bar{V}}\right]=-\nabla\bar{p}+\nabla\cdot\tau_{ij}$$

$$\tau_{ij}=(\mu+\mu_T)\left(\frac{\partial\bar{u}_i}{\partial x_j}+\frac{\partial\bar{u}_j}{\partial x_i}\right)$$

Note the omission of turbulent kinetic energy term.

- The sum of dynamic and eddy viscosity is called effective viscosity denoted as μ_e.
- In turbulent flows with high levels of turbulence the eddy viscosity is typically much larger than the dynamic viscosity.

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Boussinesq Hypothesis Based Turbulence Modeling

- Turbulence modeling approaches based on the Boussinesq Hypothesis have long been the “workhorses” of practical engineering turbulence models.
- Many models have been developed and are being used today.
- All of them, however, suffer from the same deficiency: there is no universal RANS-based model capable of predicting all types of turbulent flows under different conditions.
- Even today RANS-focused research is alive and well, and research articles are still been actively presented and their methodologies implemented in CFD codes.
- The subject of turbulence modeling is definitely worthy of a full course (or even several courses as there is definitely enough published research material for at least a few!).
- Thus, we are not going into detailed discussion of turbulence modeling and will limit ourselves to very briefly mentioning some high-level ideas to common modeling approaches.

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Turbulence Modeling Using the Boussinesq Approach

- Any modeling approach which gives us the eddy viscosity can be a possible closure for the RANS equations.
- Accordingly, we can develop formulas which are entirely algebraic, or construct models based on transport equations of turbulence-related variables.
- Algebraic models are known as zero equation models. If we solve one or more turbulence transport equations, then we refer to the models as one, two, etc., equation models, based on the number of equations solved.
- We will look at how a transport equation for an important turbulence field quantity (the turbulence kinetic energy) can be developed. This equation (and related equations) can be employed in the development of more sophisticated multiple equation turbulence models.

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Prandtl Mixing Length Hypothesis

- Prandtl assumed that the eddy viscosity was related to a parameter called the "mixing length" l as follows:

$$\mu_T(x,y) = \rho l^2 \left|\frac{\partial \bar{u}}{\partial y}\right|$$

- The mixing length can be thought of as the transverse distance over which fluid parcels retain their original momentum as they are moving erratically within the turbulent boundary layer.
- Prandtl originally proposed that the mixing length was proportional to the wall distance, y

$$l = \kappa y$$

- This was later modified by Van Driest (1956) to include a damping function to improve agreement with data:

$$l = \kappa y\left(1 - e^{-y^+/A^+}\right), \quad A^+ = 26\left[1 + y\frac{dp_e/dx}{\rho v^{*2}}\right]^{-1/2}$$

- From experiments, the constant κ was determined to be about 0.4. This would later be known as the Von Karman constant.
- Despite its simplicity, this model is quite accurate for wall-bounded flows without separation under small pressure gradients.

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Zero Equation Algebraic Turbulence Models

- Algebraic turbulence models calculate eddy viscosity directly from flow variables without solving additional equations.
- These simplistic models cannot accurately model more complex general flows, but they can be useful for a subset of flows for which they are calibrated or for a quick "zero order" qualitative evaluation of the turbulent flow field.
- Two common algebraic models are Cebeci-Smith[1-2] (1967) and Baldwin-Lomax[3] (1978).
- For example, eddy viscosity formulation of Cebeci-Smith model in 2D is:

$$\mu_T = \begin{cases} \mu_{T_{inner}}, y \le y_c \\ \mu_{T_{outer}}, y > y_c \end{cases}$$

$$\mu_{T_{inner}} = \rho l^2 \left[\left(\frac{\partial \bar{u}}{\partial y} \right)^2 + \left(\frac{\partial \bar{v}}{\partial x} \right)^2 \right]$$

$$\mu_{T_{outer}} = \alpha \rho V_e \delta^* F_K,$$

$$F_K = \left[1 + 5.5 \left(\frac{y}{\delta} \right)^2 \right]^{-1}$$

$y_c = \min(y)$ where $\mu_{T_{inner}} = \mu_{T_{outer}}$

l – mixing length

F_K –Klebanoff intermittency function

[1]Smith, A.M.O. and Cebeci, T. (1967), "Numerical solution of the turbulent boundary layer equations", Douglas aircraft division report DAC 33735.

[2]Cebeci, T. and Smith, A.M.O. (1974), "Analysis of turbulent boundary layers", Academic Press

[3]Baldwin, B. S. and Lomax, H. (1978), "Thin Layer Approximation and Algebraic Model for Separated Turbulent Flows", AIAA Paper 78-257.

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Equation for Turbulent Kinetic Energy

- Most Boussinesq-based RANS models involve a transport equation for the turbulent kinetic energy (TKE).
- The general form of the unmodeled TKE equation is derived from sum of products of Navier-Stokes equations with fluctuating velocities:

$$\frac{\partial k}{\partial t} + \overline{u_j}\frac{\partial k}{\partial x_j} = -\overline{u_i' u_j'}\frac{\partial \overline{u_i}}{\partial x_j} - \nu\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}} + \frac{\partial}{\partial x_j}\left[\nu\frac{\partial k}{\partial x_j} - \frac{1}{2}\overline{u_i' u_i' u_j'} - \overline{\frac{p' u_j'}{\rho}}\right]$$

unsteady; advection; production; turbulent viscous dissipation; molecular diffusion; turbulent transport; pressure diffusion

- The right-hand-side terms are complex and require modeling.
- The turbulent dissipation term is normally denoted as ε:

$$\varepsilon = \nu\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}}$$

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One-Equation Turbulence Models

- One-equation models solve one transport equation for a turbulence quantity.
- Originally, Prandtl's one-equation model[1] solved for the TKE, and then correlated it to the eddy viscosity through the turbulent length scale. This model, however, did not show much advantage over algebraic approaches.
- A very successful and commonly used one-equation model is Spalart-Allmaras model[2] which solves for modified eddy viscosity:

$$\frac{\partial \tilde{\nu}}{\partial t} + \bar{u}_j \frac{\partial \tilde{\nu}}{\partial x_j} = G_\nu + \frac{1}{\sigma_{\tilde{\nu}}}\left[\frac{\partial}{\partial x_j}\left((\nu + \tilde{\nu})\frac{\partial \tilde{\nu}}{\partial x_j}\right) + C_{b2}\left(\frac{\partial \tilde{\nu}}{\partial x_j}\right)^2\right] - Y_\nu$$

production of $\tilde{\nu}$ destruction of $\tilde{\nu}$

$$\mu_T = \rho \tilde{\nu} f_{\nu 1}$$

- Expressions for production and destruction terms, and for the model constants not shown here, are somewhat involved. They can be found in Ref [2].
- The Spalart-Allmaras model was originally developed for wall-bounded flows in aerospace applications, and it proved capable of accurately predicting boundary layers under adverse pressure gradients.

[1] Wilcox, D.C. (2004) "Turbulence Modeling for CFD."

[2] Spalart, P. R. and Shur, M. L. (1997) "On the Sensitization of Turbulence Models to Rotation and Curvature", Aerospace Sci. Tech. 1(5). 297–302.

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Two-Equation Turbulence Models

- Two-equation models represent the majority of RANS based turbulent models. They solve two equations for turbulent kinetic energy k and turbulent dissipation rate ε, or its variant. Sometimes instead of the dissipation rate, an equation for turbulence length scale l is solved.
- These equations are heavily modeled using many ad hoc assumptions and empirical correlations of modeling constants.

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Two-Equation Turbulence Models: $k-\varepsilon$ Model

- $k-\varepsilon$ model[1] of Launder and Spalding is a classical example of a two-equation model:

$$\rho\left(\frac{\partial k}{\partial t}+\bar{u}_j\frac{\partial k}{\partial x_j}\right)=\frac{\partial}{\partial x_j}\left[\left(\mu+\frac{\mu_T}{\sigma_k}\right)\frac{\partial k}{\partial x_j}\right]+\mu_T\frac{\partial \bar{u}_i}{\partial x_j}\left(\frac{\partial \bar{u}_i}{\partial x_j}+\frac{\partial \bar{u}_j}{\partial x_i}\right)-\rho\varepsilon$$

$$\rho\left(\frac{\partial \varepsilon}{\partial t}+\bar{u}_j\frac{\partial \varepsilon}{\partial x_j}\right)=\frac{\partial}{\partial x_j}\left[\left(\mu+\frac{\mu_T}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_j}\right]+C_{1\varepsilon}\mu_T\frac{\varepsilon}{k}\frac{\partial \bar{u}_i}{\partial x_j}\left(\frac{\partial \bar{u}_i}{\partial x_j}+\frac{\partial \bar{u}_j}{\partial x_i}\right)-\rho C_{2\varepsilon}\frac{\varepsilon^2}{k}$$

- Eddy viscosity is constructed from dimensional considerations for μ_T and ε:

$$\left.\begin{array}{l}\varepsilon\propto k^{3/2}/l\\ \mu_T\propto \rho k^{3/2}/l\end{array}\right\}\Rightarrow\quad \mu_T=\rho C_\mu\frac{k^2}{\varepsilon}$$

- σ_k and σ_ε are effective "Prantdl numbers" relating eddy diffusion of k and ε to the momentum equation eddy viscosity.
- Models constants are correlated to benchmark experimental data for wall-bounded flows.
- The original $k-\varepsilon$ model suffers from many deficiencies, and nowadays more modern and reliable two-equation models are favored over this "veteran" of turbulence modeling. It is still used for teaching purposes to illustrate basic turbulence modeling principals.

[1] Launder, B. E., and Spalding, D. B. (1972) "Lectures in Mathematical Models of Turbulence," Academic Press, London, England.

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Reynolds Stress Models

- An alternative to the Boussinesq-based methods is the approach of deriving and solving equations for individual Reynolds stresses. The idea is to preserve as much “exactness” of the original RANS equations as possible to avoid many modeling assumptions and manipulations of models based on eddy viscosity.
- Reynolds stress models (RSMs) are not constrained by the Boussinesq assumption of isotropic turbulence, and they are capable of predicting complex turbulent flows where eddy viscosity models may fail, e. g., swirl-dominated flows or problems with secondary flows.
- Derivation of RSMs is quite complex, and it involves multiplying the exact momentum equations for the fluctuations and applying Reynolds averaging.
- This exact process, however, still gives rise to several unknown terms which require modeling to close the equations.

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Summary

- We have discussed two approaches to closing RANS equations: the Boussinesq Hypothesis to model the Reynolds stresses and direct modeling of Reynolds stresses.
- Models based on the Boussinesq hypothesis are most common in predicting turbulence in practical engineering problems, while RSMs are used in specific situations where eddy viscosity models fail.
- Turbulence modeling is a very vast and complex topic, and more detailed discussion of various approaches and models will be reserved for a dedicated course.

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