

Potential Flows

Simple Approximations of Fluid Flows – Lesson 3



/ Potential Flow

- A potential flow is a simplified model of fluid motion which assumes that the flow is:
 - **Inviscid** (viscous effects are unimportant)
 - **Incompressible** (constant density)
 - **Irrotational** (vorticity is zero)
 - **Force field** is conservative
 - **Isothermal** (uniform temperature, so the energy equation is not required)
- When are the potential flow assumptions reasonable?
 - Flow of liquids and gases at low speeds (compressibility effect is negligible)
 - Viscous effects confined to thin regions near walls
 - Upstream flow is vorticity-free (i.e., smooth and undisturbed)
 - Body forces are due to gravity or can be neglected entirely
- We also typically assume the flow to be **steady-state**, although this is not strictly required for the potential flow model.

Governing Equations for Potential Flow

- Let us begin with the governing equations for a steady-state inviscid, incompressible, irrotational flow with conservative body forces derived in the previous lesson:

$$\nabla \cdot \vec{V} = 0$$

Continuity equation

$$p + \rho \frac{V^2}{2} + B = C$$

Momentum equation given by Bernoulli's formula

- For an irrotational flow field, the velocity can be expressed in terms of the velocity potential ϕ as:

$$\vec{V} = \nabla \phi$$

- And the continuity equation reduced to the **Laplace's equation**, one of the most famous and well-studied equations of mathematical physics:

$$\nabla^2 \phi = 0$$

/ Governing Equations for Potential Flow (cont.)

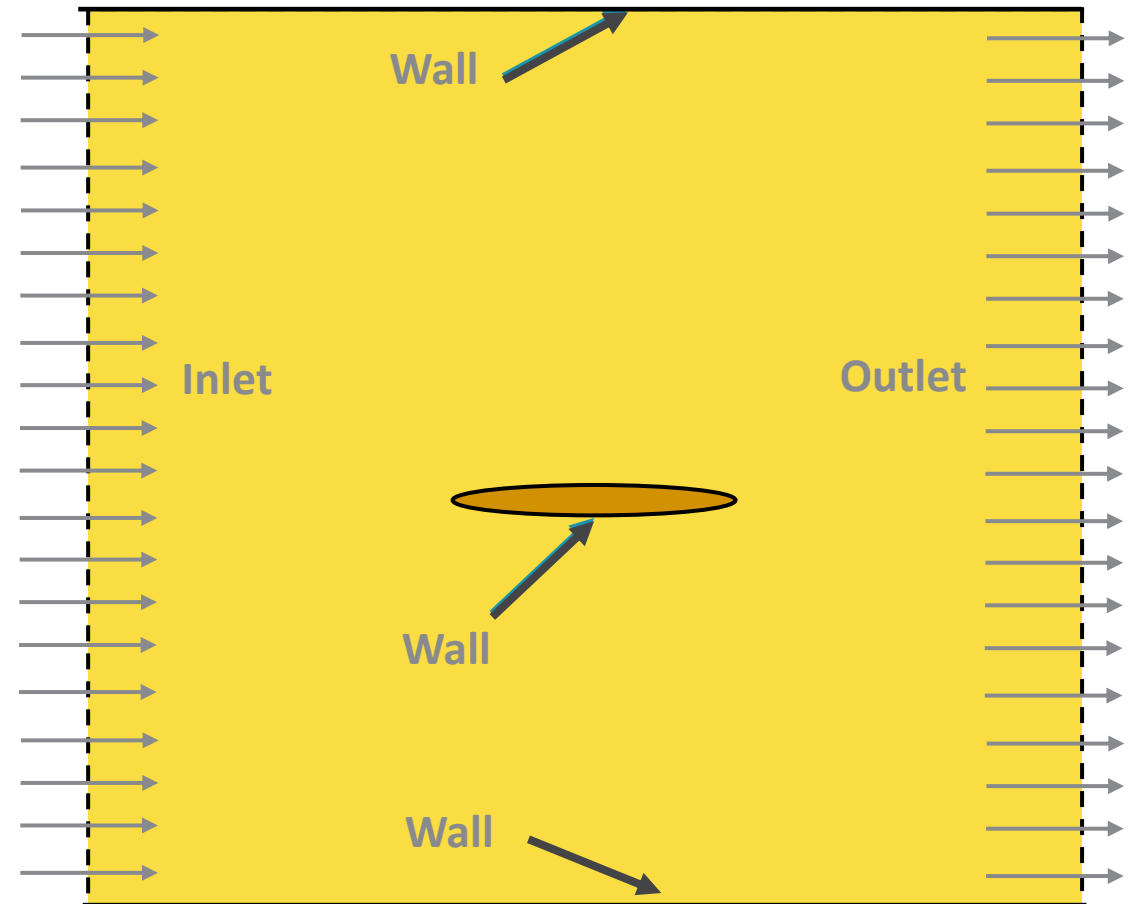
- There are many solutions to Laplace's equation, and since Laplace's equation is linear, we can add solutions together to form a new solution. This technique is called **superposition**.
- There are also many well-known approximate solution techniques for Laplace's equation that utilize computers for complicated geometries and boundary conditions.
- Once the potential field is found from the Laplace equation, the velocity field is calculated by differentiating the potential, and then pressure distribution is computed from the momentum (Bernoulli's) equation using the known velocity field.

Boundary Conditions for Potential Flow (cont.)

- In order to solve Laplace's Equation for ϕ , we need to (1) define a region of interest where we will solve the equations and (2) define the mathematical conditions at the boundaries of this region.
- A problem governed by the Laplace equation is an elliptic boundary value problem requiring information about the potential on all the domain boundaries.
- From the theory of Partial Differential Equations (PDEs), there are two types of boundary conditions which can be specified for an elliptic PDE:
 - Prescribed value potential (Dirichlet condition)
 - Prescribed value of derivative of potential (Neumann condition)
- Different types of boundary conditions can be specified on different boundaries.
- Neumann condition alone does not yield a unique solution, which differs by an arbitrary constant from other solutions.
- Specification of Dirichlet condition on at least one boundary is required for the solution to be unique.

Boundary Conditions for Potential Flow (cont.)

- Physics of fluid flows provides natural specification of boundary conditions at different boundaries:
 - **Wall Boundaries** – we assume no flow penetrates a wall (although a velocity component tangent to the wall is permitted), and Neumann condition specified at the walls.
 - **Inlet and Outlet Boundaries** – usually velocity values are known or can be estimated at inlets and outlets where the flow enters and leaves the domain. Thus, the Dirichlet boundary condition is prescribed at inlets and outlets for potential flow solutions.



Wall Boundary Conditions for Velocity Potential

- At wall boundaries, we assume **no flow penetrates a wall**. Thus, in terms of the wall unit normal, the normal velocity at the wall is set to zero as follows:

$$(V_n)_{wall} = (\nabla\phi \cdot \hat{n})_{wall} = \left(\frac{\partial\phi}{\partial n}\right)_{wall} = 0$$

- Note that the normal derivative boundary condition for ϕ is sufficient for the solution of Laplace's equation and the tangent derivative (which defines the local velocity tangent to the wall) is determined from the solution.

Inlet / Outlet Boundary Conditions for Velocity Potential

- For inlet and outlet boundaries, we need to set the velocity components such that the gradient of the velocity potential equals the freestream velocity vector.

$$u_{\infty} = \left(\frac{\partial \phi}{\partial x} \right)_{\infty} \quad v_{\infty} = \left(\frac{\partial \phi}{\partial y} \right)_{\infty} \quad w_{\infty} = \left(\frac{\partial \phi}{\partial z} \right)_{\infty}$$

- An equivalent way of prescribing the velocities is to use a linear variation of ϕ which satisfies the gradient requirement above.

$$\phi_{\infty}(x, y, z) = u_{\infty}x + v_{\infty}y + w_{\infty}z$$

- Note that this is an exact solution for a uniform flow and satisfies Laplace's equation for all x, y, z .

/ The Stream Function

- Here we will introduce the concept of a stream function which is helpful in deriving analytical solutions for potential flows in two dimensions.
- The stream function $\psi(x, y)$ in 2D is defined as:

$$u = -\frac{\partial\psi}{\partial y}$$

$$v = \frac{\partial\psi}{\partial x}$$

- The stream function derives its name from the fact that its iso-lines are nothing more than streamlines tangent to the velocity vector, which can be proven by the mathematical derivation presented on the next slide.
- While the definition of a stream function is constant along flow streamlines, in three dimensions it is possible that the form becomes complex and thus loses its usefulness for deriving analytical solutions.
- Thus, our discussion will be limited to 2D stream functions.

The Stream Function and Streamlines

- To prove that the stream function represents streamlines of a 2D flow, recall that the definition of a 2D streamline is:

$$v dx = u dy$$

- Now, let us consider the differential of ψ along a streamline. If this is an iso-line, then the differential $d\psi$ is zero. Expanding $d\psi$ using the chain rule gives

$$d\psi = \frac{\partial\psi}{\partial x} dx + \frac{\partial\psi}{\partial y} dy = 0$$

- Using the definitions of the velocity components in terms of derivatives of ψ and substituting into the above yields

$$v dx - u dy = 0$$

which is the definition of the 2D streamline.

/ The Stream Function and Potential Flow

- Like the velocity potential, the stream function for potential flows satisfies Laplace's equation.
 - Substitution of the velocity expressed in terms of the stream function into the irrotationality condition, gives the Laplace equation for ψ

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

- Stream function contours (lines of constant ψ) are orthogonal to the velocity potential contours (lines of constant ϕ).

Wall Boundary Conditions for Stream Function

- As we did previously with the velocity potential, we assume no flow penetrates a wall. However, an iso-value of the stream function is a streamline, across which there can be no flow. Therefore, a wall is a boundary of constant ψ :

$$\psi_{wall} = constant$$

- Note that the actual value of the constant is arbitrary but should be consistent with other boundary conditions for the problem at hand.

Inlet and Outlet Boundary Conditions for Stream Function

- At inlet and outlet boundaries, we can set the velocity components using the derivative definitions provided earlier:

$$u_{\infty} = \left(-\frac{\partial\psi}{\partial y}\right)_{\infty} \quad v_{\infty} = \left(\frac{\partial\psi}{\partial x}\right)_{\infty}$$

- An equivalent way of prescribing the velocities is to use a linear variation of ψ which satisfies the above relations:

$$\psi_{\infty}(x, y) = -u_{\infty}y + v_{\infty}x$$

- Note that this is an **exact solution** for a uniform flow and satisfies Laplace's equation for all x and y .

/ Summary

- This lesson has considered potential flow model formulation.
- We have examined the equations for general potential flow derived in terms of the velocity potential and showed that it takes the form of a Laplace equation.
- In addition, we also introduced a special form of the potential flow equation in two dimensions, expressed in terms of the stream function, which also reduced to the Laplace equation for the stream function.
- Finally, we introduced boundary conditions required for solving potential flows
- In the next lesson we will look at some exact solutions for potential flows.

 **Ansys**

