Other Boundary Layer Solutions and 3D Layers

Laminar Boundary Layer Theory – Lesson 5

Ansys

General Form of 2D Boundary Layer with Non-Constant Mean Flow

- The boundary layer equations can be extended to the case of non-constant mean flow $V_\infty = V_\infty\,(x,t)$:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \left(\frac{\partial V_\infty}{\partial t} + V_\infty\frac{\partial V_\infty}{\partial x}\right) + \frac{1}{\rho}\frac{\partial \tau}{\partial y}$$

- Here τ is the shear stress which for laminar flows has the familiar form:

$$\tau = \mu\frac{\partial u}{\partial y}$$

Note: The above form of the boundary layer equation also holds for turbulent boundary layers, but with a different expression for τ.

Karman Momentum Integral Equation

- This general boundary layer equation can be integrated to derive integral relationships. Multiplying continuity by $(\mathrm{u} - V_\infty)$ and subtracting from the momentum equation, the integral form is:

$$\frac{\partial}{\partial t}\int_0^\infty (V_\infty - u)dy + \underbrace{\frac{\partial}{\partial x}\int_0^\infty u(V_\infty - u)dy}_{\text{momentum thickness}} + \frac{\partial V_\infty}{\partial x}\underbrace{\int_0^\infty (V_\infty - u)dy}_{\text{displacement thickness}} - V_\infty v_w = \frac{\tau_w}{\rho}$$

$v_w\,(x)$ - for cases of porous wall with injection / suction

- This equation can be rewritten in terms of displacement and momentum thicknesses as:

$$\frac{1}{V_\infty^2}\frac{\partial}{\partial t}(V_\infty \delta^*) + \frac{\partial \theta}{\partial x} + (2\theta + \delta^*)\frac{1}{V_\infty}\frac{\partial V_\infty}{\partial x} - \frac{v_w}{V_\infty} = \frac{\tau_w}{\rho V_\infty^2} = \frac{C_f}{2}$$

- Assuming steady flow and non-porous wall, the relation reduces to:

$$\frac{d\theta}{dx} + (2 + H)\frac{\theta}{V_\infty}\frac{dV_\infty}{dx} = \frac{C_f}{2}, \qquad H = \frac{\delta^*}{\theta}$$

- This form of the Karman integral relation will come handy in our analysis of turbulent boundary layers.

Correlation Method of Thwaites

- Thwaites (1949) proposed the following correlation method for the Karman integral relation.

  Multiplying the Karman relation by $V_\infty \theta/\nu$ and defining a parameter λ as, $\lambda = \left(\theta^2/\nu\right)\left(dV_\infty/dx\right)$, gives:

$$V_\infty \frac{d}{dx}\left(\frac{\lambda}{dV_\infty/dx}\right) = 2\left[\frac{\tau_w \theta}{\mu V_\infty} - \lambda(2+H)\right]$$

where:

$$\tau_w \theta/\mu V_\infty = S(\lambda)$$ shear correlation

$$H = H(\lambda)$$ shape-factor correlation

- This equation can be rewritten as:

$$V_\infty \frac{d}{dx}\left(\frac{\lambda}{dV_\infty/dx}\right) \approx 2[S(\lambda) - \lambda(2+H)] = F(\lambda)$$

- Thwaites examined the entire collection of experimental results and found that there is a simple linear fit:

$$F(\lambda) \approx \underbrace{0.45}_{a} - \underbrace{6}_{b}\lambda$$

- The solution of the ODE is then:

$$\frac{\theta^2}{\nu} = a V_\infty^{-b}\left(\int_{x_0}^{x} V_\infty^{b-1} dx + C\right)$$

where the constant $C = 0$ to avoid $\theta \to \infty$ when x_0 is a stagnation point

Correlation Method of Thwaites

- Thus, Thwaites correlation predicts $\theta(x)$ very accurately within $\pm 5\%$ for favorable and mild adverse pressure gradients and $\pm$15% near separation points for laminar boundary layers by the simple quadratic relation:

$$\theta^2 \approx \frac{0.45\nu}{V_\infty^6}\int_0^x V_\infty^5 dx$$

- Shear stress and displacement thickness are:

$$\tau_w = \frac{\mu V_\infty}{\theta} S(\lambda)$$

and $S(\lambda)$ is given by a simple correlation

$$S(\lambda) \approx (\lambda + 0.09)^{0.62}$$

$$\delta^* = \theta H(\lambda)$$

and $S(\lambda)$ can be fitted, after some effort, by a polynomial:

$$H(\lambda) \approx 2.0 + 4.14z - 83.5z^2 + 854z^3 - 3337z^4 + 4576z^5$$

$$z = 0.25 - \lambda$$

The Falkner-Skan Equation

- Following in the footsteps of Blasius, a more general similarity solution approach was developed by V. M Falkner and S. W. Skan in 1930 for flows over wedge-shaped geometries.

- They generalized the Blasius solution to variable freestream velocity:

$$u(x,y) = V_\infty(x) f'(\eta)$$

and found that a similarity solution exists if the freestream velocity has a power-law distribution:

$$V_\infty(x) = Cx^m$$

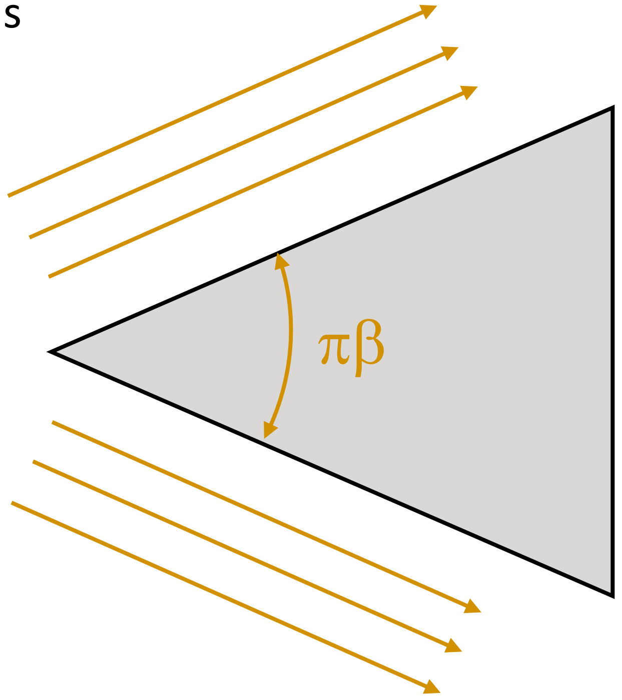
$$\eta = y\sqrt{\frac{m+1}{2}\frac{V_\infty(x)}{\nu x}}$$

- The solution is given by the following ODE and boundary conditions:

$$f''' + ff'' + \beta[1-(f')^2] = 0$$

$$\beta = \frac{2m}{m+1}$$

$$f(0) = f'(0) = 0$$
$$f'(\infty) = 1$$



Special Cases:
β = 0 → m = 0 (flat plate)
β = 1 → m = 1 (vertical plate)

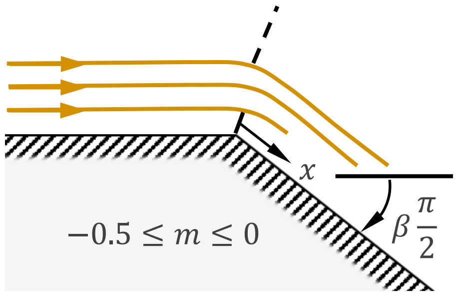
The Falkner-Skan Equation

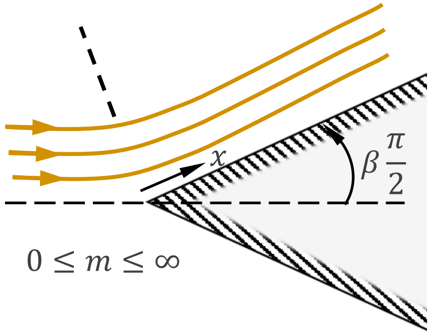
- Since the x-momentum equation now retains the pressure gradient, the external pressure gradient can be calculated using the prescribed velocity above and Bernoulli's equation.

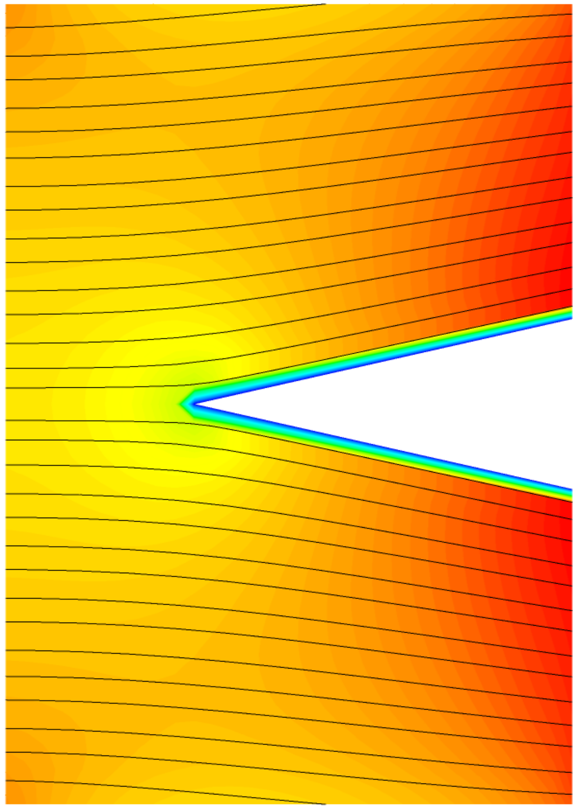
- The solution of the Falkner-Skan equation proceeds similarly to Blasius (using identical boundary conditions for f), except that a numerical integration method for the ODE must be used, as outlined earlier for the Blasius equation.

- $m < 0$ corresponds to adverse pressure gradient up to the separation point ($m = -0.09043$, $\beta = -0.19884$) and the solution becomes nonphysical past this point.

- $m > 0$ corresponds to favorable pressure gradients, and the solution exists up to $m = 0$.

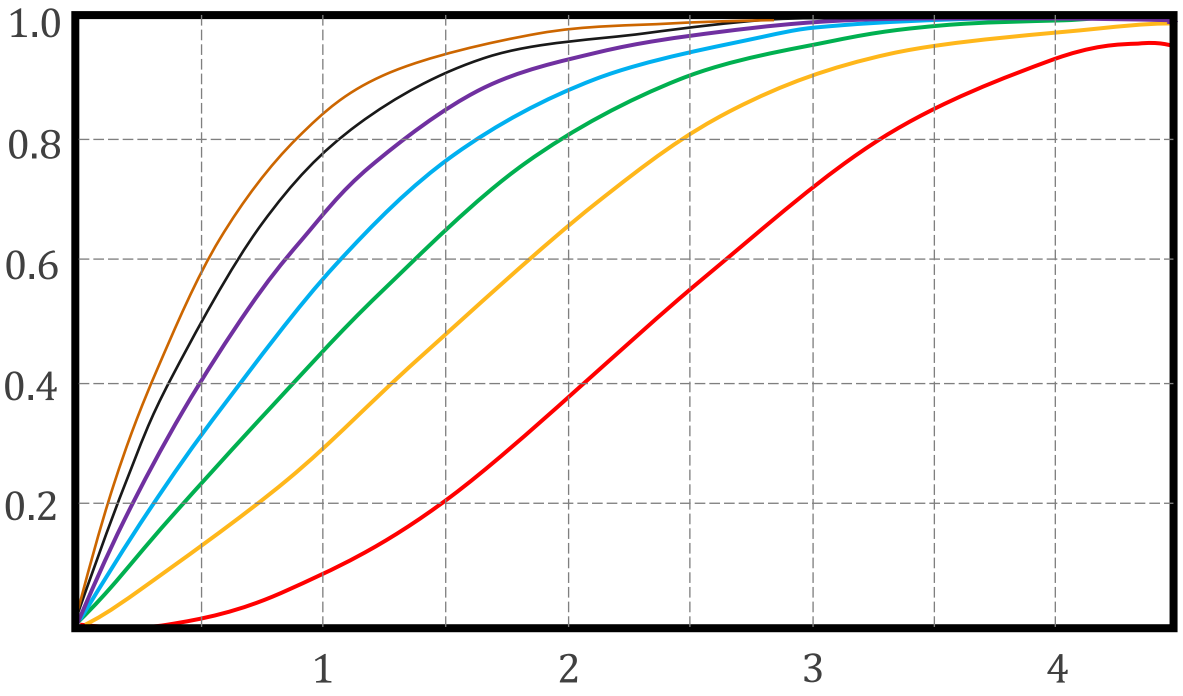






Velocity contours and streamlines for a flow over a wedge.

The Falkner-Skan Profiles for Selected Values of m



$$f'(\eta) = \frac{u}{V_\infty}$$

$$\eta = y\sqrt{\frac{m+1}{2}\frac{V_\infty(x)}{\nu x}}$$

m	β
-0.091	-0.199
-0.0654	-0.14
0	0
$1/9$	0.2
$1/3$	0.5
1	1
4	1.6

Velocity distribution in the laminar boundary layer of the wedge flow

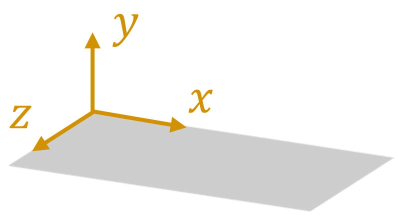
Three-Dimensional Boundary Layers

- The two-dimensional approach of Prandtl can be extended to 3D laminar boundary layers, and corresponding equations can be derived. For example for a 3D layer over a flat plate aligned with $x - z$ plane, the boundary layer equations become:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = U_\infty \frac{\partial U_\infty}{\partial x} + W_\infty \frac{\partial U_\infty}{\partial z} + \nu \frac{\partial^2 u}{\partial y^2}$$

$$u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = U_\infty \frac{\partial W_\infty}{\partial x} + W_\infty \frac{\partial W_\infty}{\partial z} + \nu \frac{\partial^2 w}{\partial y^2}$$



$U_\infty(x,z)$, $W_\infty(x,z)$ Known freestream velocity components

- In general, a Blasius type analytical solution is difficult or even impossible to obtain.

- These are parabolic equations in x, and they can be solved numerically by marching the solution downstream. Even though the solution process requires computer coding, it will be simpler than a full numerical solution of Navier-Stokes equation.

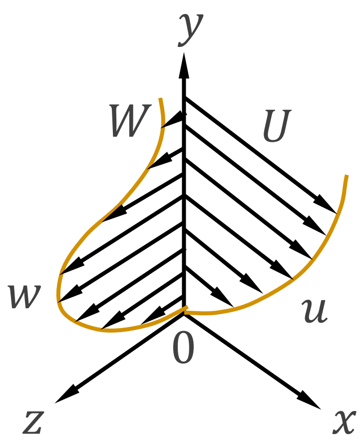
Secondary Flows in Three-Dimensional Layers

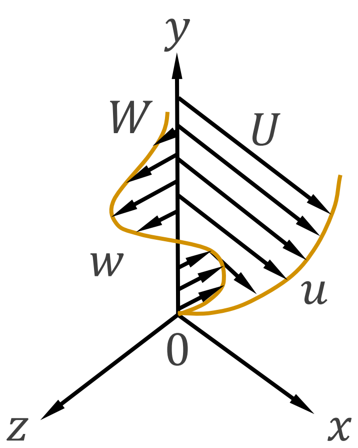
- If U_∞ is locally aligned with the mainstream direction and W_∞ is zero, then $w\ (x, z)$ component represents crossflow or secondary flow.

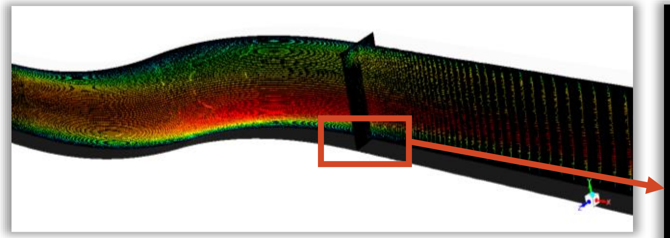
- This secondary flow depends on crossflow pressure gradients and streamline curvature.

- Examples of secondary boundary layer flows include:
  - Aircraft swept-back wings where the boundary layer near the trailing edge moved outward along the wing axis.
  - Cross-flows generated by pressure gradients on turbomachinery blades and propellers.

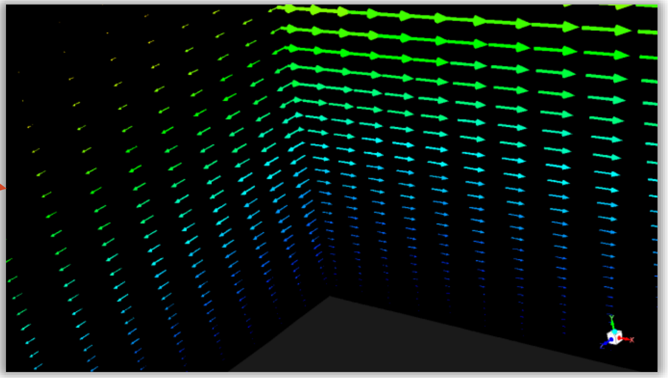
- Thus, the effects of secondary flows must be understood, either by analytical, empirical or numerical means, as they change dynamics of boundary layers and affect design decisions.







Secondary flow in a duct after an s-bend



Friedrichs' Boundary Layer Model

- One inherent deficiency of the boundary layer problem is its singular perturbation nature. Recall the boundary layer momentum equation:

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = \cancel{\frac{1}{Re} \frac{\partial^2 u^*}{\partial y^{*2}}}$$

0 as $Re \rightarrow \infty$

- In the limit of the Reynolds number becoming large, $Re \rightarrow \infty$, the second-derivative term vanishes, the equations reduce to first-order and the non-slip condition on the wall can no longer be satisfied.

- This makes the classical boundary layer approach, strictly speaking, non-physical for very large Re as no-slip condition still holds at those Re values.

- Friedrichs (1942) attempted to correct this deficiency of the classical boundary layer theory, which was later expanded by Van Dyke (1964), to give asymptotic methodology to resolve it.

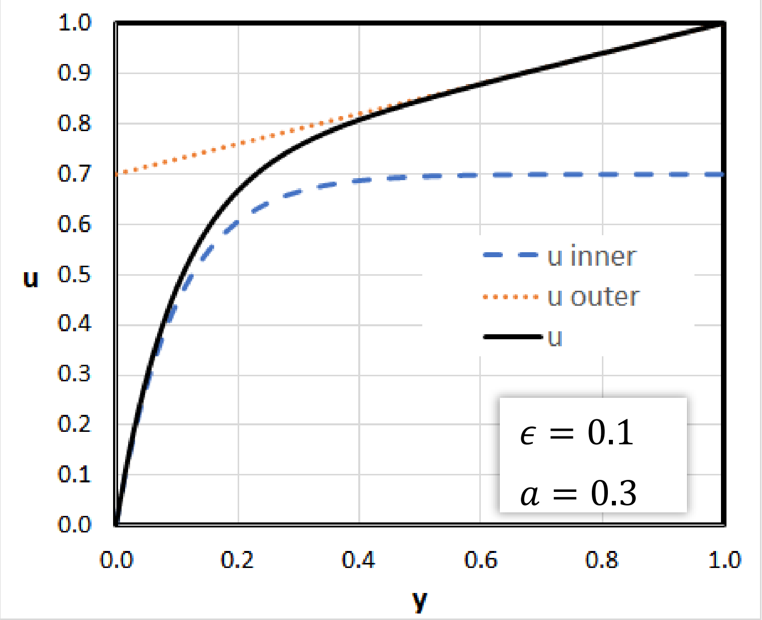
Friedrichs' Boundary Layer Model (cont.)

- The boundary layer momentum equation can be roughly approximated by the following ODE:

$$\epsilon \frac{d^2u}{dy^2} + \frac{du}{dy} = a, \qquad \epsilon \ll 1$$
$$u(0) = 0, \qquad u(1) = 1$$

- The boundary layer solution can be written in terms of inner and outer parts satisfying respective boundary conditions at the wall (inner) and freestream (outer).

$$u_{outer} = (1-a) + ay$$
$$u_{inner} = (1-a)(1-e^{-y/\epsilon})$$



- They then can be blended into a composite function representing the entire regions:

$$u = (1-a)\left(1-e^{-y/\epsilon}\right) + ay$$

Matched Asymptotic Expansions

- Van Dyke generalized this first-order procedure into an approach of matching inner and outer expansions of any order.

- This methodology was used to correct for:
  - Leading edge effects at low Reynolds number
  - Trailing edge effect

- For example, second order asymptotic correction to the flat-plate boundary layer leads to arousal of an additional term in the drag coefficient expression:

$$C_D(L) = \underbrace{\frac{1.338}{\sqrt{Re_L}}}_{\text{Blasius solution}} + \underbrace{\frac{2.326}{Re_L}}_{\text{2nd order correction}}$$

- This correction extends prediction of drag over flat plate to lower Reynold numbers, $1 < Re < 1000$, albeit slightly under-predicting it in this range.

- The methodology of matching solutions plays an important role in analyzing turbulent boundary layers.

Summary

- In this lesson we discussed a general form of a 2D flat-plate layer, and covered the Faulkner-Skan solution of boundary layer flows over wedge shapes which was obtained by the approach similar to that of Blasius.

- We also took a brief look at three-dimensional boundary layers and commented on complexity arising in 3D layers due to the development of secondary flows.

- Finally, we considered Friedrichs' approach to modeling boundary layers based on matching inner and outer solutions, which provides a useful basis for a more general method of matched asymptotic expansions which will be helpful in examining turbulent boundary layers.

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