

The Buckingham-Pi Theorem

Dimensional Analysis and Similarity – Lesson 3



Buckingham-Pi Theorem

- Can we determine the relevant dimensionless parameters for a given flow problem without recourse to the governing equations? Yes! The method is known as the **Buckingham-Pi Theorem**, which can be stated as follows:

💡 *For a problem described by a set of equations based on k physical variables we can define n dimensionless parameters (Π_1, \dots, Π_n) , where n is the difference between the number (k) of physical variables and the number (p) of fundamental quantities that describes them.*

- The procedure for determining the dimensionless parameters will be illustrated for a simple flow problem on the next slides.

Parameters and Dimensions

Variable	Dimensions
Length	L
Area	L^2
Volume	L^3
Velocity	L / T
Acceleration	L / T^2
Momentum	ML / T
Force	ML / T^2
Energy/Work	ML^2 / T^2
Power	ML^2 / T^3
Viscosity	M / LT
Pressure	M / LT^2
Density	M / L^3

L = length
 M = mass
 T = time

Application of the Buckingham-Pi Theorem to Pipe Flow

- Let's consider flow in a pipe. We determine the pressure drop (ΔP) depends on five variables: length (l), diameter (D), density (ρ), velocity (V), and viscosity (μ). From these variables we have three basic physical dimensions (length, mass, time). Therefore, we expect $6 - 3 = 3$ dimensionless Π groups.

- We can form the first dimensionless group by considering ΔP , V , D and ρ as: $D^a V^b \rho^c \Delta P^d$

which has the following physical dimensions:

$$[L]^a [LT^{-1}]^b [ML^{-3}]^c [ML^{-1}T^{-2}]^d = L^{a+b-3c-d} M^{c+d} T^{-b-2d}$$

- For this group to be non-dimensional, exponents of the dimensions must be zero:

$$\begin{aligned}a + b - 3c - d &= 0 \\c + d &= 0 \\-b - 2d &= 0\end{aligned}$$

- From this systems of three linear equations, by expressing a , b and c in terms of d , we obtain

$$c = -d, \quad b = -2d, \quad a = 0, \text{ which then leads to the first dimensionless group:}$$

$$\Pi_1 = \frac{\Delta P}{\rho V^2}$$

- The denominator, multiplied by $\frac{1}{2}$, represents a common pressure scale called dynamic pressure, $\frac{1}{2}\rho V^2$.

Application of the Buckingham-Pi Theorem to Pipe Flow (cont.)

- The second dimensionless group can be formed by considering V , D , ρ and μ : $D^a V^b \rho^c \mu^d$
- Performing analysis similar to that used in the derivation of Π_1 , we obtain the second dimensionless group:

$$\Pi_2 = \frac{\mu}{\rho D V}$$

which is nothing more than the inverse Reynolds number, $1/Re$.

- The third dimensionless group is trivial, and it is simply the ratio of two length scales L and D :

$$\Pi_3 = \frac{L}{D}$$

/ Helpful Remarks about Buckingham-Pi Theorem

- The Buckingham-Pi theorem describes a method for computing dimensionless parameters from given variables, even if the exact set of governing equations is unknown.
- Buckingham's theorem provides a methodology to calculate sets of dimensionless parameters, but it says nothing about their physical significance.
- If any two variables have the same dimension, then their ratio will be one of the non-dimensional groups.
- Calculated non-dimensional groups can be in any functional form which may be different from customarily used expressions, e.g., an inverse function.

Summary of Important Dimensionless Groups

Parameter	Definition
Reynolds Number	$Re = \frac{\rho VL}{\mu}$
Froude Number	$Fr = \frac{V}{\sqrt{gL}}$
Prandtl Number	$Pr = \frac{C_p \mu}{k}$
Mach Number	$M = \frac{V}{c}$
Brinkman number	$Br = \frac{V^2 \mu}{kT}$
Peclet Number	$Pe = RePr$
Eckert Number	$Ec = \frac{Br}{Pr}$

/ Summary

- The Buckingham-Pi Theorem provides a mathematically formal basis for deriving non-dimensional groups for any physical problem and it is not limited to fluid dynamics.
- This formal approach uses mathematics and does not assign any physical meaning or significance to non-dimensional groups.
- Understanding the physics of the underlying problem is required to interpret and correctly apply non-dimensional groups in a subsequent analysis of this physical model.

 **Ansys**

