Transmission Line Theory

Sources

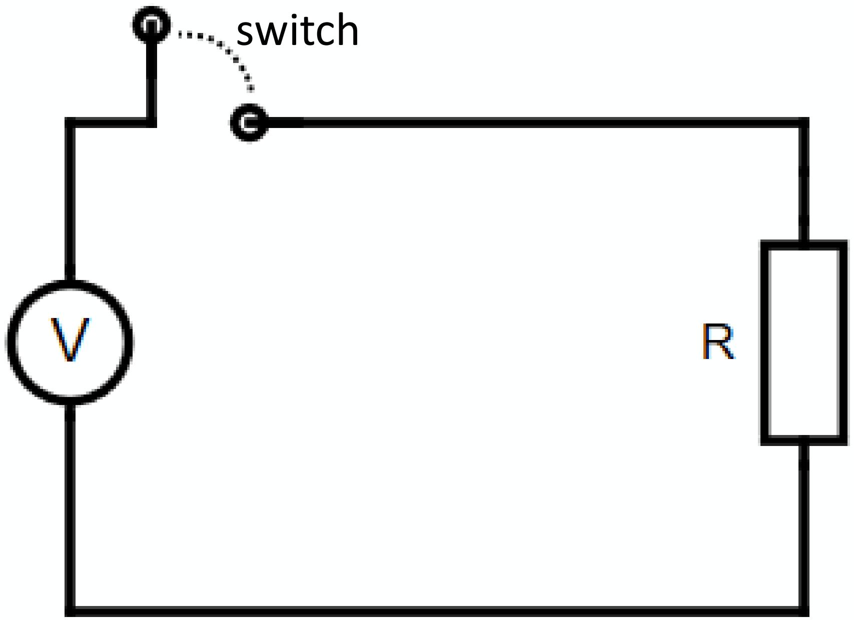
The material presented herein is from the following sources:

"Elements of Electromagnetics," by Matthew N.O Sadiku, 5th ed. (2010)
"Engineering Electromagnetics," by Nathan Ida, 3rd ed. (2015)
"Microwave Engineering," by David Pozar, 4th ed. (2012)

Why talk about transmission lines?

Electrically large circuits require more advanced analysis techniques than those taught for electrically-small systems.

Consider the following circuit diagram:



By conventional circuit theory (under the electrically-small-system assumption), closing the switch will instantly result in the voltage *V* appearing across the resistor *R*.

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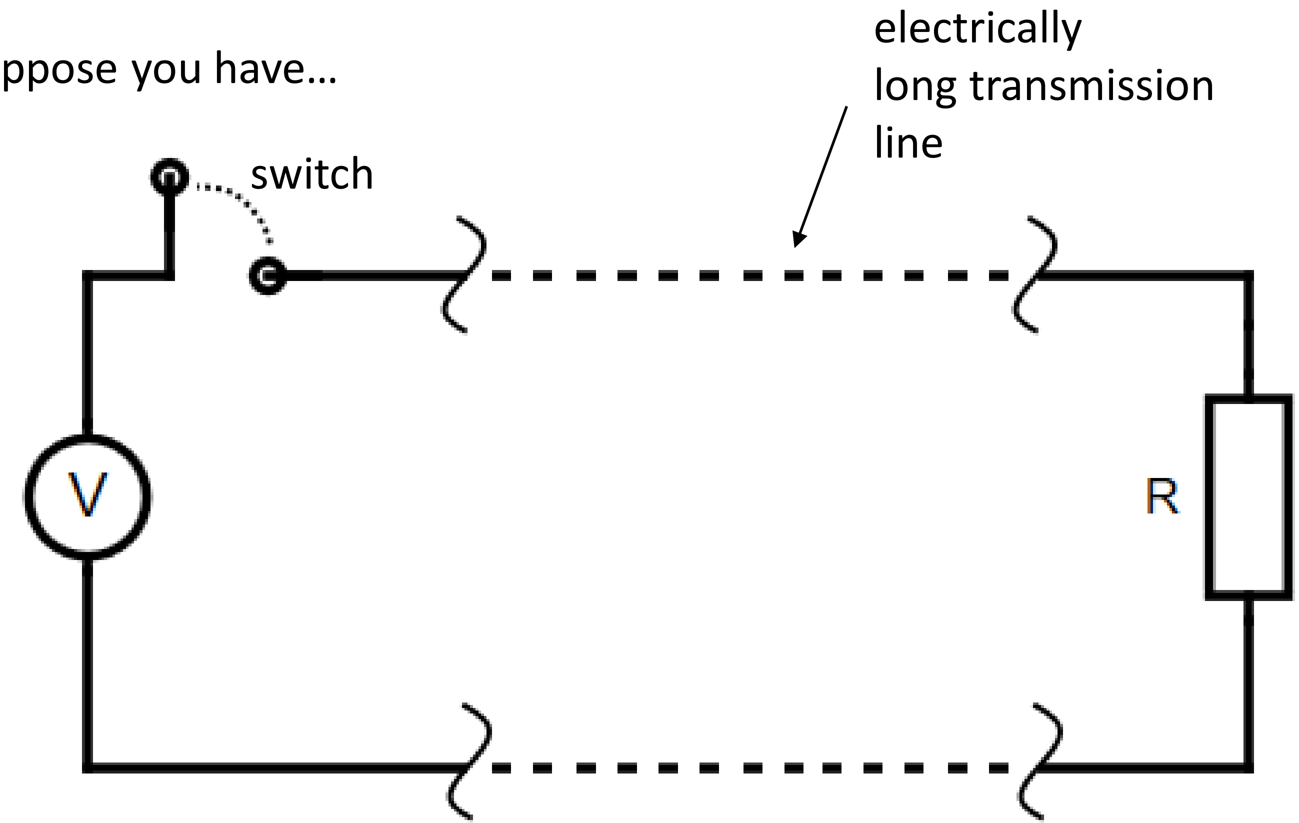
But suppose you have…

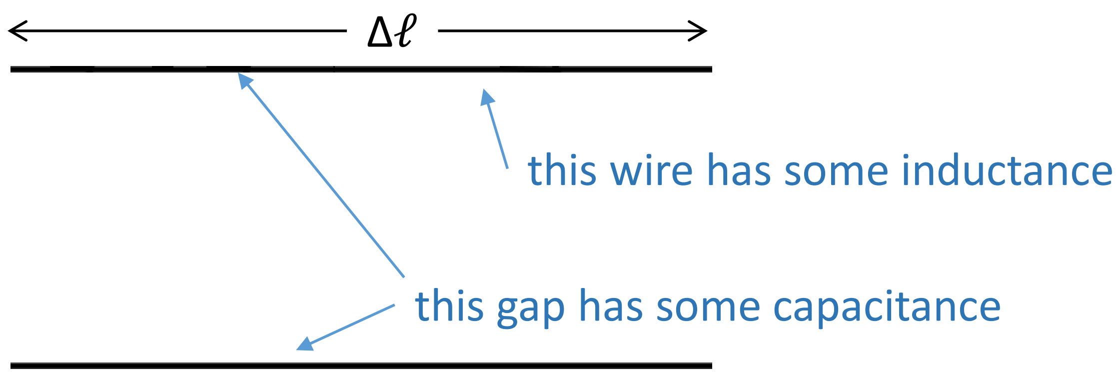


If the switch is far away from the load, then closing the switch will *not* instantaneously result in the voltage *V* appearing at the load. Rather, there will be a delay as the signal is transmitted down the line.

The Telegrapher's Equations

Electrically large circuits require more advanced analysis techniques than those taught for electrically-small systems.

Let us look closely at a section of transmission line, of length $\Delta\ell$, where $\Delta\ell$ is electrically small ($\Delta\ell << \lambda$)



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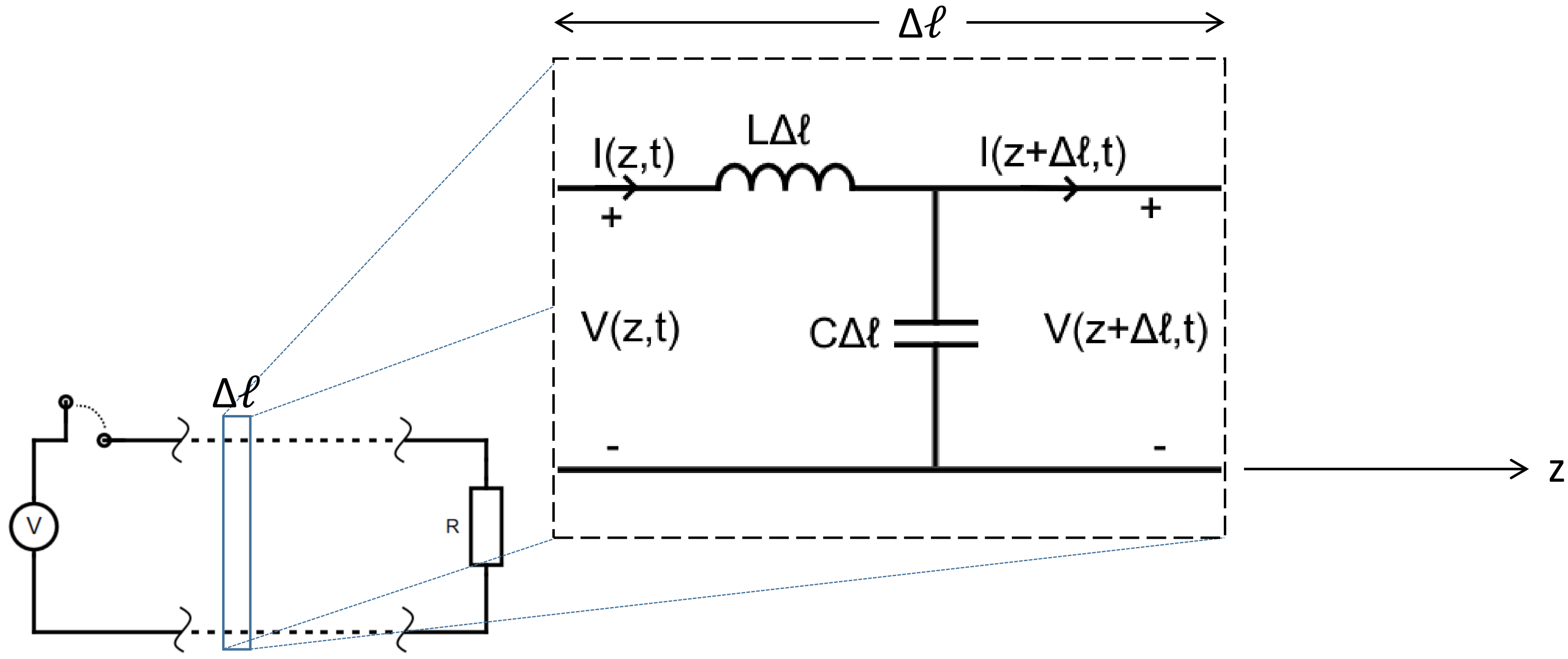
So, each piece of the long line has an inductance-per-length *L* and a capacitance-per-length *C*, which may be modeled as shown above.



Note that the total capacitance of this length Δℓ of transmission line is CΔℓ, and the total inductance is LΔℓ.

The Telegrapher's Equations

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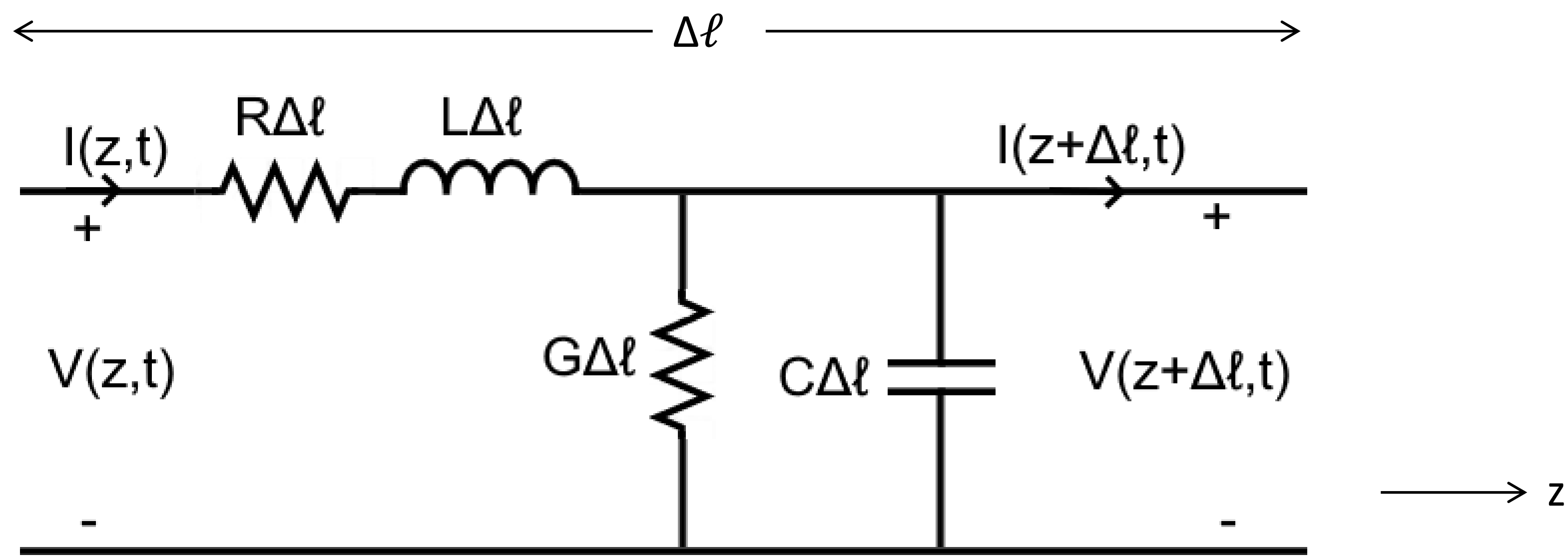
In the most general case, the line may also have resistive losses, which can be added to the model as:

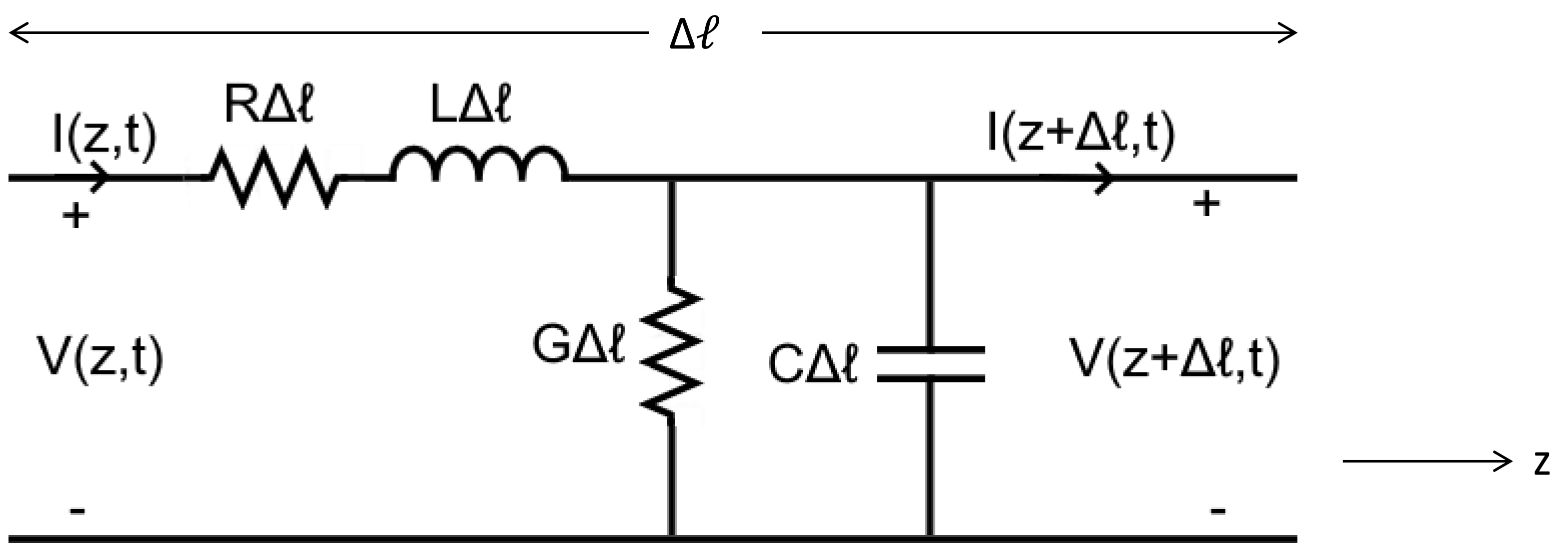


Note that here, the total series resistance of this length $\Delta\ell$ of transmission line is R$\Delta\ell$, and the total shunt conductance is G$\Delta\ell$.

The Telegrapher's Equations

Electrically large circuits require more advanced analysis techniques than those taught for electrically-small systems.

This system has input voltage V(z,t), input current I(z,t), output voltage V(z+$\Delta\ell$,t) and output current I(z+$\Delta\ell$,t).



Since we have assumed that $\Delta\ell$ is electrically small, we can analyze this circuit using normal circuit analysis.

The Telegrapher's Equations

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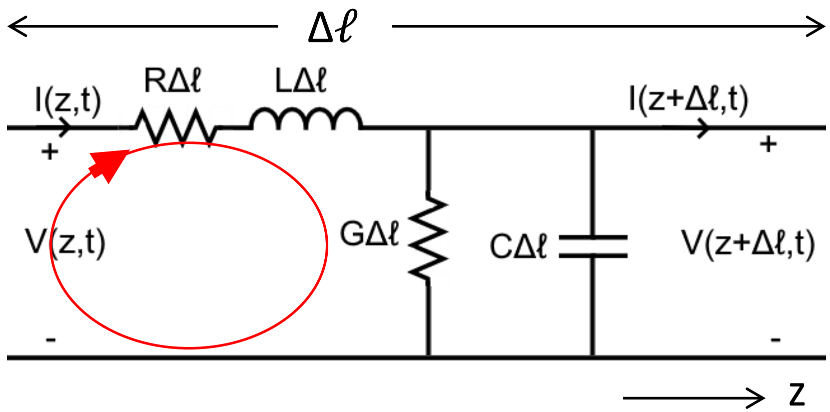

Using Kirchoff's voltage law, we can write:

$$V(z,t) - R\Delta\ell I(z,t) - L\Delta\ell \frac{dI(z,t)}{dt} - V(z+\Delta\ell,t) = 0$$

which can be rearranged to show:

$$\frac{V(z+\Delta\ell,t) - V(z,t)}{\Delta\ell} + L\frac{dI(z,t)}{dt} + RI(z,t) = 0$$

and, taking the limit as $\Delta\ell \to 0$, this becomes the spatial derivative:

$$\boxed{\frac{dV(z,t)}{dz} + L\frac{dI(z,t)}{dt} + RI(z,t) = 0}$$

The Telegrapher's Equations

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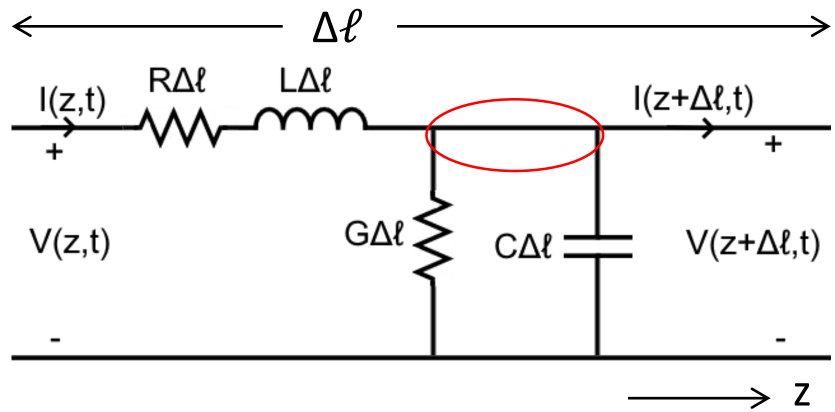


Similarly, by Kirchoff's current law:

$$I(z,t) - G\Delta\ell V(z+\Delta\ell,t) - C\Delta\ell \frac{dV(z+\Delta\ell,t)}{dt} - I(z+\Delta\ell,t) = 0$$

which can be rearranged to show:

$$\frac{I(z+\Delta\ell,t) - I(z,t)}{\Delta\ell} + C\frac{dV(z+\Delta\ell,t)}{dt} + GV(z+\Delta\ell,t) = 0$$

and, taking the limit as $\Delta\ell \rightarrow 0$, this becomes the spatial derivative:

$$\boxed{\frac{dI(z,t)}{dz} + C\frac{dV(z,t)}{dt} + GV(z,t) = 0}$$

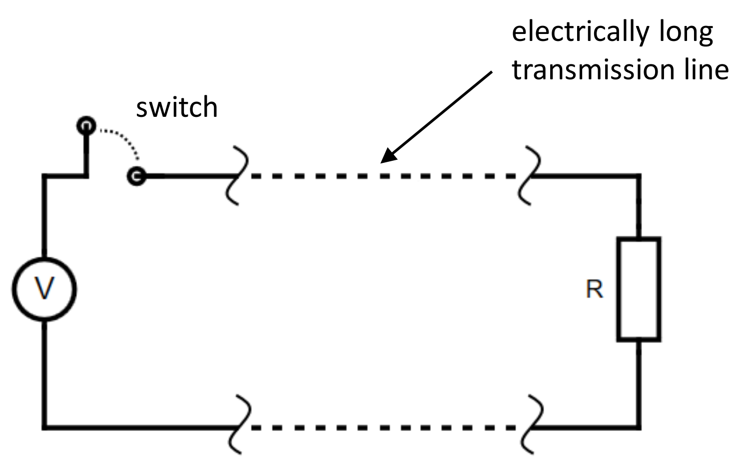
The Telegrapher's Equations

Electrically large circuits require more advanced analysis techniques than those taught for electrically-small systems.

$$\text{(1)}\quad \boxed{\frac{dV(z,t)}{dz} + L\frac{dI(z,t)}{dt} + RI(z,t) = 0}$$

$$\text{(2)}\quad \boxed{\frac{dI(z,t)}{dz} + C\frac{dV(z,t)}{dt} + GV(z,t) = 0}$$

These are the **telegrapher's equations**, which relate voltage and current along a transmission line.



The Wave Equations

The Telegrapher's Equations may be used to derive the wave equations for voltage and current along a transmission line.

A **wave equation** relates a quantity's second derivative **in time** to its second derivative **in space**.

Current and voltage on a transmission line may be described using wave equations, which can be derived from the telegrapher's equations as follows:

1 $$\boxed{\frac{dV(z,t)}{dz} = -L\frac{dI(z,t)}{dt} - RI(z,t)}$$

2 $$\boxed{\frac{dI(z,t)}{dz} = -C\frac{dV(z,t)}{dt} - GV(z,t)}$$

$\frac{d}{dt}$ 2 $\rightarrow$ $$\frac{d^2I(z,t)}{dzdt} = -C\frac{d^2V(z,t)}{dt^2} - G\frac{dV(z,t)}{dt}$$

$\frac{d}{dz}$ 1 $\rightarrow$ $$\frac{d^2V(z,t)}{dz^2} = -L\frac{d^2I(z,t)}{dzdt} - R\frac{dI(z,t)}{dz}$$

$$\frac{d^2V(z,t)}{dz^2} = -L\left[-C\frac{d^2V(z,t)}{dt^2} - G\frac{dV(z,t)}{dt}\right] - R\left[-C\frac{dV(z,t)}{dt} - GV(z,t)\right]$$

$$\boxed{\frac{d^2V(z,t)}{dz^2} = LC\frac{d^2V(z,t)}{dt^2} + (LG+RC)\frac{dV(z,t)}{dt} + RGV(z,t)}$$

$\rightarrow$ Wave equation for **voltage**

The Wave Equations

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$\frac{d}{dz}$ 2 $\rightarrow$ $$\frac{d^2I(z,t)}{dz^2} = -C\frac{d^2V(z,t)}{dzdt} - G\frac{dV(z,t)}{dz}$$

$$\frac{d^2I(z,t)}{dz^2} = -C\left[-L\frac{d^2I(z,t)}{dt^2} - R\frac{dI(z,t)}{dt}\right] - G\left[-L\frac{dI(z,t)}{dt} - RI(z,t)\right]$$

$$\boxed{\frac{d^2I(z,t)}{dz^2} = LC\frac{d^2I(z,t)}{dt^2} + (LG+RC)\frac{dI(z,t)}{dt} + RGI(z,t)}$$

$\rightarrow$ Wave equation for **current**

The Wave Equations

The Telegrapher's Equations may be used to derive the wave equations for voltage and current along a transmission line.

$$\frac{d^2V(z,t)}{dz^2} = LC\frac{d^2V(z,t)}{dt^2} + (LG + RC)\frac{dV(z,t)}{dt} + RGV(z,t)$$

This is the wave equation for **voltage**, which relates its second derivative in **time** to its second derivative in **space**.

$$\frac{d^2I(z,t)}{dz^2} = LC\frac{d^2I(z,t)}{dt^2} + (LG + RC)\frac{dI(z,t)}{dt} + RGI(z,t)$$

This is the wave equation for **current**, which relates its second derivative in **time** to its second derivative in **space**.

Note that the wave equations for voltage and current are **identical**. These differential equations will also have identical solutions.

Lossless Transmission Lines

For a lossless line, the telegrapher's equations and wave equations can be simplified.

On a lossless transmission line, the series resistance vanishes (R=0), and the shunt conductance vanishes (G=0). For this case, the telegrapher's equations become:

$$\boxed{\frac{dV(z,t)}{dz} = -L\frac{dI(z,t)}{dt}} \qquad \boxed{\frac{dI(z,t)}{dz} = -C\frac{dV(z,t)}{dt}}$$

The Lossless Telegrapher's Equations

and the wave equations become:

$$\boxed{\frac{d^2V(z,t)}{dz^2} = LC\frac{d^2V(z,t)}{dt^2}} \qquad \boxed{\frac{d^2I(z,t)}{dz^2} = LC\frac{d^2I(z,t)}{dt^2}}$$

The Lossless Wave Equations

Wave Equation Solutions

Current and voltage will propagate on the line according to the solutions to the wave equations.

We will assume that both voltage and current are time-harmonic (~ $e^{j\omega t}$). In other words, we assume that:

$$V(z,t) = V(z)e^{j\omega t}$$

$$I(z,t) = I(z)e^{j\omega t}$$

This allows us to write the wave equations as:

$$\frac{d^2V(z)}{dz^2}e^{j\omega t} = -\omega^2 LCV(z)e^{j\omega t} + j\omega(LG + RC)V(z)e^{j\omega t} + RGV(z)e^{j\omega t}$$

$$\frac{d^2I(z)}{dz^2}e^{j\omega t} = -\omega^2 LCI(z)e^{j\omega t} + j\omega(LG + RC)I(z)e^{j\omega t} + RGI(z)e^{j\omega t}$$

Or, by rearranging:

$$\frac{d^2V(z)}{dz^2} = V(z)[RG + j\omega(LG + RC) - \omega^2 LC]$$

$$\frac{d^2I(z)}{dz^2} = I(z)[RG + j\omega(LG + RC) - \omega^2 LC]$$

Propagation Constant

We will pause here to define the complex propagation constant

The complex propagation constant γ is defined as:

$$\gamma^2 = RG + j\omega(LG + RC) - \omega^2 LC$$

or,

$$\gamma = \sqrt{RG + j\omega(LG + RC) - \omega^2 LC}$$

We will also define the **attenuation constant** α and the **lossless propagation constant** β as the real and imaginary parts of γ, respectively.

$$\gamma = \alpha + j\beta$$

Note that, **in the lossless case,**

$$\begin{aligned} \gamma &= j\omega\sqrt{LC} \\ &= \alpha + j\beta \\ &= j\beta \end{aligned}$$

where

$$\alpha = 0$$

and $\beta = \omega\sqrt{LC}$

Wave Equation Solutions

Current and voltage will propagate on the line according to the solutions to the wave equations.

Using our definition of the complex propagation constant, we can rewrite the wave equations as:

$$\frac{d^2V(z)}{dz^2} = \gamma^2 V(z)$$

$$\frac{d^2I(z)}{dz^2} = \gamma^2 I(z)$$

which are differential equations with solutions of either sines and cosines or complex exponentials. We will choose to use complex exponentials.

$$\boxed{\begin{aligned} V(z) &= \mathrm{V_o^+}\mathrm{e}^{-\gamma z} + \mathrm{V_o^-}\mathrm{e}^{+\gamma z} \\ I(z) &= \mathrm{I_o^+}\mathrm{e}^{-\gamma z} + \mathrm{I_o^-}\mathrm{e}^{+\gamma z} \end{aligned}}$$

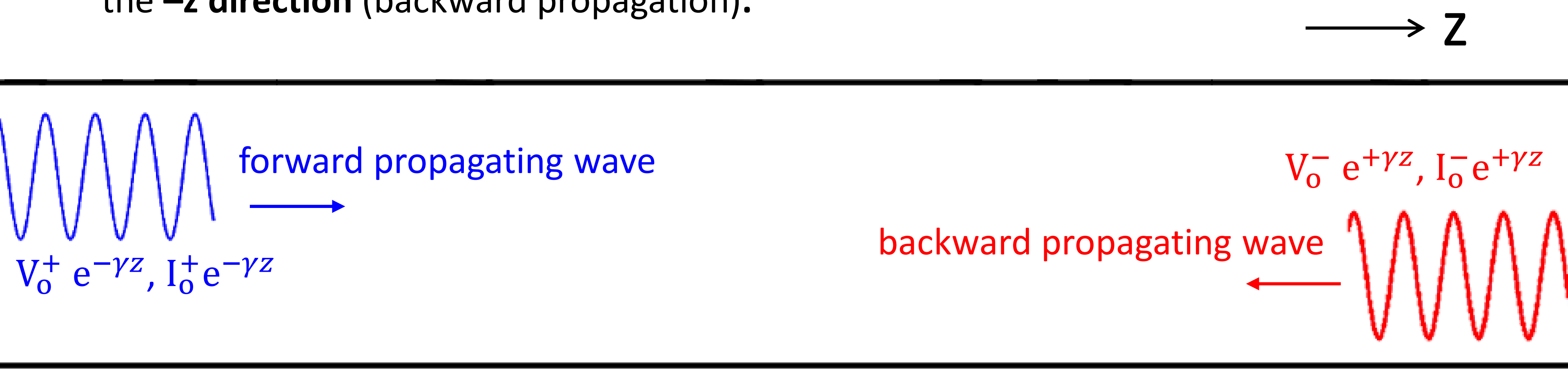
where $\mathrm{V_o^+}$, $\mathrm{V_o^-}$, $\mathrm{I_o^+}$, and $\mathrm{I_o^-}$ are variables corresponding to magnitude.

Wave Equation Solutions

Current and voltage will propagate on the line according to the solutions to the wave equations.

$$\boxed{V(z) = \mathrm{V_o^+} \mathrm{e}^{-\gamma z} + \mathrm{V_o^-} \mathrm{e}^{+\gamma z}} \qquad \boxed{I(z) = \mathrm{I_o^+} \mathrm{e}^{-\gamma z} + \mathrm{I_o^-} \mathrm{e}^{+\gamma z}}$$

Each of these solutions consists of two terms. The terms including $\boldsymbol{e^{-\gamma z}}$ propagate in the **+z direction** (forward propagation), and the terms including $\boldsymbol{e^{+\gamma z}}$ propagate in the **–z direction** (backward propagation).



The total voltage on the transmission line is the sum of the forward-propagating part and the backward propagating part.

The total current is similarly the sum of the forward and backward propagating terms.

Wave Equation Solutions

Current and voltage will propagate on the line according to the solutions to the wave equations.

Recall: we said earlier that

$$V(z,t) = V(z)e^{j\omega t}$$
$$I(z,t) = I(z)e^{j\omega t}$$

so our total solutions for voltage and current are:

$$V(z,t) = \mathrm{V_o^+ e^{-\gamma z}} e^{j\omega t} + \mathrm{V_o^- e^{+\gamma z}} e^{j\omega t}$$
$$I(z,t) = \mathrm{I_o^+ e^{-\gamma z}} e^{j\omega t} + \mathrm{I_o^- e^{+\gamma z}} e^{j\omega t}$$

but this is commonly written in phasor form, which suppresses the harmonic term:

$$V(z,t) = \mathrm{V_o^+ e^{-\gamma z} + V_o^- e^{+\gamma z}} = \mathrm{V_\circ^+ e^{-\alpha z} e^{-j\beta z} + V_\circ^- e^{+\alpha z} e^{+j\beta z}}$$
$$I(z,t) = \mathrm{I_o^+ e^{-\gamma z} + I_o^- e^{+\gamma z}} = \mathrm{I_\circ^+ e^{-\alpha z} e^{-j\beta z} + I_\circ^- e^{+\alpha z} e^{+j\beta z}}$$

or it may be written equivalently in the time domain:

$$V(z,t) = \mathrm{V_o^+ e^{-\alpha z}} \cos(\omega t - \beta z) + \mathrm{V_o^- e^{+\alpha z}} \cos(\omega t + \beta z)$$
$$I(z,t) = \mathrm{I_o^+ e^{-\alpha z}} \cos(\omega t - \beta z) + \mathrm{I_o^- e^{+\alpha z}} \cos(\omega t + \beta z)$$

Characteristic Impedance

The ratio of voltage to current at any point along a transmission line is fixed by the characteristics of the line.

Let us look at just the forward-propagating components of voltage and current, in phasor form:

$$V(z,t) = \mathrm{V_o^+} \mathrm{e}^{-\gamma z}$$

$$I(z,t) = \mathrm{I_o^+} \mathrm{e}^{-\gamma z}$$

and recall the first of the telegrapher's equations:

$$\frac{dV(z,t)}{dz} = -L\frac{dI(z,t)}{dt} - RI(z,t)$$

We can combine these equations as follows:

$$\frac{d}{dz}[\mathrm{V_o^+} \mathrm{e}^{-\gamma z}] = -L\frac{d}{dt}[\mathrm{I_o^+} \mathrm{e}^{-\gamma z}] - R[\mathrm{I_o^+} \mathrm{e}^{-\gamma z}]$$

$$-\gamma \mathrm{V_o^+} \mathrm{e}^{-\gamma z} = -j\omega L \mathrm{I_o^+} \mathrm{e}^{-\gamma z} - R\mathrm{I_o^+} \mathrm{e}^{-\gamma z}$$

which can be rearranged to give the characteristic impedance, defined by:

$$Z_o = \frac{\mathrm{V_o^+}}{\mathrm{I_o^+}} = \frac{j\omega L + R}{\gamma} = \frac{j\omega L + R}{\sqrt{RG + j\omega(LG + RC) - \omega^2 LC}} = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

Wave Impedance

The ratio of voltage to current at any point along a transmission line is fixed by the characteristics of the line.

This is the characteristic impedance of the line, given in terms of its per-length resistance, inductance, conductance, and capacitance.

$$Z_o = \frac{\mathrm{V_o^+}}{\mathrm{I_o^+}} = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

Note that, if the line is lossless, this becomes:

$$Z_o = \sqrt{L/C}$$

Note that the characteristic impedance of a lossless line is purely real.

The characteristic impedance is the ratio of forward voltage to forward current. The same derivation may be performed in terms of the backward voltage and backward current, to show:

$$Z_o = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \frac{forward\ voltage}{forward\ current} = -\frac{backward\ voltage}{backward\ current}$$

Note the negative sign of the backward terms

Wave Properties

Let us pause here to summarize the wave properties we've encountered so far…

Angular Frequency: $\omega = 2\pi f$

Complex Propagation Constant: $\gamma = \sqrt{RG + j\omega(LG + RC) - \omega^2 LC} = \alpha + j\beta$

Attenuation Constant: $\alpha = Re\{\gamma\}$

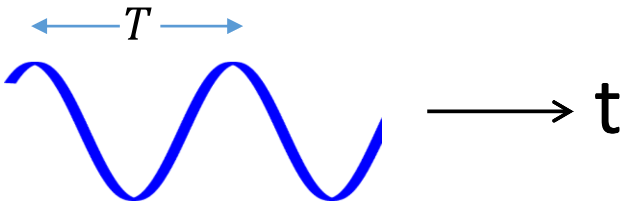
Lossless Propagation Constant: $\beta = Im\{\gamma\} = \dfrac{2\pi}{\lambda}$

Characteristic Impedance: $Z_o = \sqrt{\dfrac{R + j\omega L}{G + j\omega C}}$

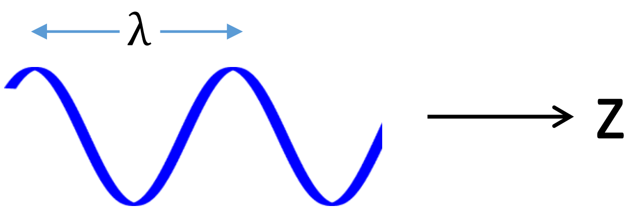
Wave Properties

Let us pause here to summarize the wave properties we've encountered so far…

Period: $T = 1/f$ = time in seconds between two peaks of a wave at a given location in space



Wavelength: λ = distance in meters between two peaks of a wave at a given instant in time



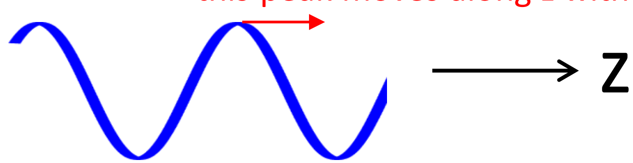
Wave Properties

Additionally, let us define group and phase velocity.

Phase velocity: $v_{ph} = \dfrac{\omega}{\beta}$

$=$ velocity of the phase front of the wave

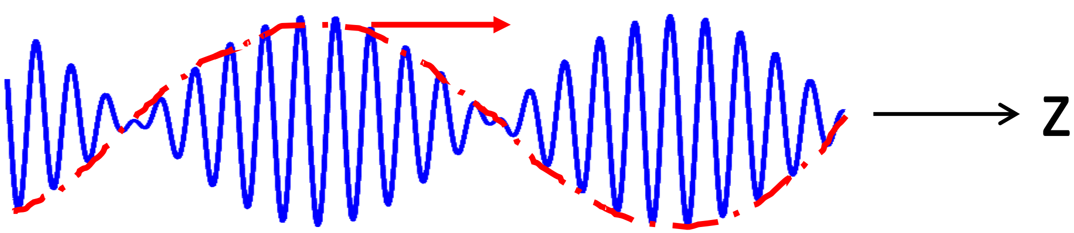
this peak moves along z with velocity v_{ph}



Group velocity: $v_g = \dfrac{\delta\omega}{\delta\beta}$ (this definition holds for **narrowband** signals)

$=$ velocity of the **envelope** of the wave

the envelope peak (dashed red) moves along z with velocity v_{ph}



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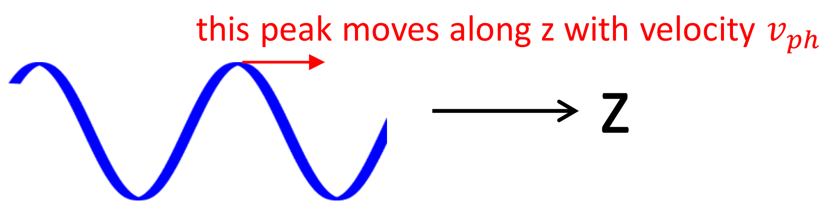

Note that in a lossy medium, the phase velocity is a function of frequency. This leads to **dispersion**, where a wave packet containing a range of frequencies (such as a square wave) input at one end will lose its shape over the length of the line, as the various frequencies travel at different rates.

In a **lossless** medium (R=G=0), the frequency-dependence disappears, and we are left with:

$$\boxed{v_{ph} = \frac{1}{\sqrt{LC}}}$$

Wave Properties

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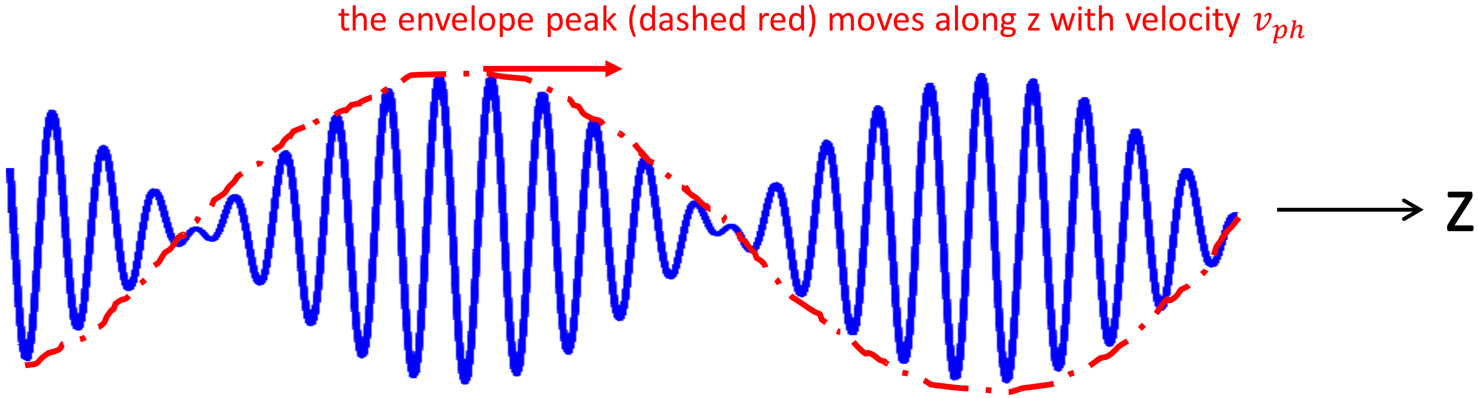
Group velocity: $v_g = \dfrac{\delta\omega}{\delta\beta}$ (this definition holds for **narrowband** signals)

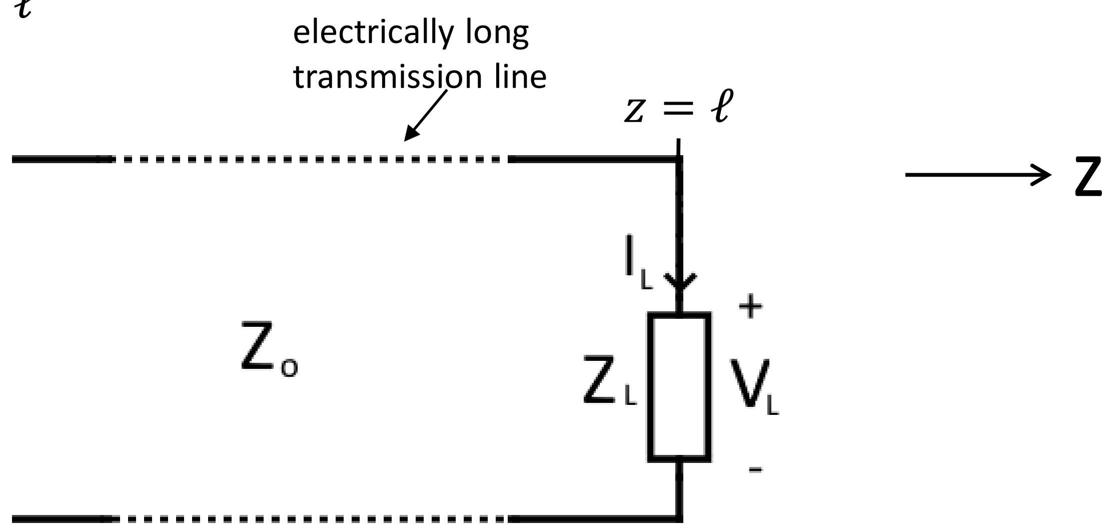
= velocity of the **envelope** of the wave



Note that if the wave consists of a single frequency, the group and phase velocities are equal.

Energy Transfer to Load

Let us look at what happens when a lossless transmission line is used to drive a load at location $z = \ell$



At the load end, the ratio of total voltage to total current must conform to Ohm's Law

$$Z_L = \frac{V_L}{I_L} = \frac{V(\ell)}{I(\ell)} = \frac{V_o^+ e^{-j\beta\ell} + V_o^- e^{+j\beta\ell}}{I_o^+ e^{-j\beta\ell} + I_o^- e^{+j\beta\ell}}$$

On the line, the total voltage and total voltage consist of forward traveling and backward traveling components, which are related by:

$$Z_o = \frac{V_o^+}{I_o^+} = -\frac{V_o^-}{I_o^-}$$

Energy Transfer to Load

Let us look at what happens when a lossless transmission line is used to drive a load at location $z = \ell$

$$\mathrm{Z}_L = \frac{V_L}{I_L} = \frac{V(\ell)}{I(\ell)} = \frac{\mathrm{V_o^+ e^{-j\beta\ell} + V_o^- e^{+j\beta\ell}}}{\mathrm{I_o^+ e^{-j\beta\ell} + I_o^- e^{+j\beta\ell}}}$$

$$\mathrm{Z_o} = \frac{V_o^+}{I_o^+} = -\frac{V_o^-}{I_o^-}$$

These two relationships may be rearranged to show that:

$$\frac{V_o^-}{V_o^+} = \frac{Z_L \mathrm{e}^{-j\beta\ell} - Z_o \mathrm{e}^{-j\beta\ell}}{Z_L \mathrm{e}^{+j\beta\ell} + Z_o \mathrm{e}^{+j\beta\ell}}$$

And we can arbitrarily assign $z = \ell = 0$ as the zero location for the z-axis. This leads to the following definition of the reflection coefficient:

$$\Gamma = \frac{V_o^-}{V_o^+} = \frac{Z_L - Z_o}{Z_L + Z_o}$$

Reflection Coefficient

What does the reflection coefficient mean?

$$\boxed{\Gamma = \frac{V_o^-}{V_o^+} = \frac{Z_L - Z_o}{Z_L + Z_o}}$$

If a transmission line is driven with input voltage V_o^+, the signal will travel down the line to the load. Once it reaches the load, there are two possible routes it may follow. The signal may flow into the load, or it may reflect back toward the source. The reflection coefficient gives the ratio of the reflected voltage to the input voltage.

Consider three special cases.

1) $Z_L = Z_o$ The is the perfect match scenario. In this case, $\Gamma = 0$. All the signal flows into the load.

2) $Z_L \to \infty$ The is the open circuit scenario. In this case, $\Gamma = 1$. All the signal reflects back toward the source.

3) $Z_L = 0$ The is the short circuit scenario. In this case, $\Gamma = -1$. All the signal reflects back toward the source, with a phase shift of $180°$.

Reflection Coefficient

$$\boxed{\Gamma = \frac{V_o^-}{V_o^+} = \frac{Z_L - Z_o}{Z_L + Z_o}}$$

Note: On a lossless line, the characteristic impedance is positive and purely real. If the load is passive ($Re\{Z_L\} \geq 0$), then the magnitude of the reflection coefficient is between zero and one ($0 \leq |\Gamma| \leq 1$)

We will define the standing wave ratio (SWR) on the line as:

$$\boxed{\mathrm{SWR} = \frac{1 + |\Gamma|}{1 - |\Gamma|}}$$

and the return loss (RL) as:

$$\boxed{\mathrm{RL} = -20 \log_{10}|\Gamma| \quad \text{, dB}}$$

Note that Γ, SWR, and RL are all various expressions of reflection.

Reflections on a Transmission Line

Reflections on a transmission line are caused by ***impedance discontinuities***

Consider:

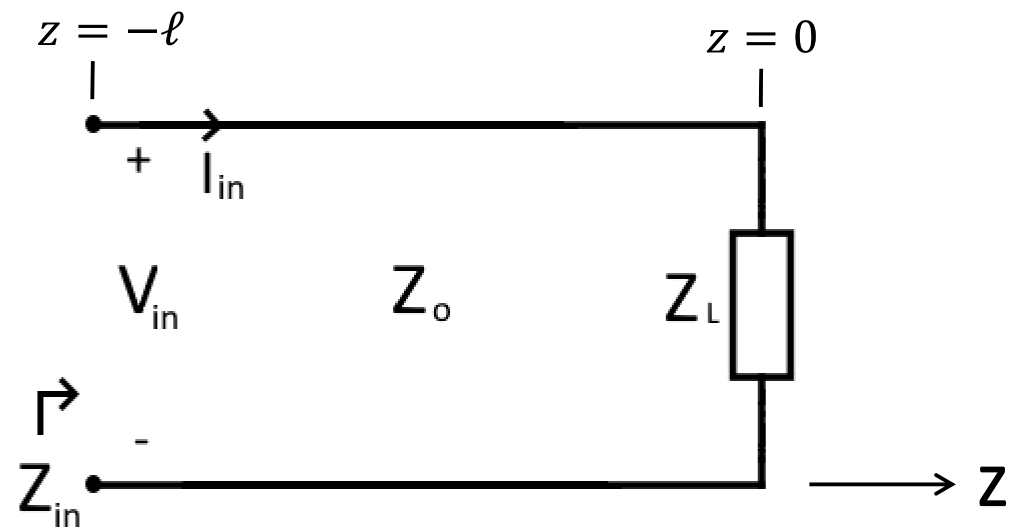
$$\boxed{\Gamma = \frac{V_o^-}{V_o^+} = \frac{Z_L - Z_o}{Z_L + Z_o}}$$

The reflection is nonzero whenever $Z_L \neq Z_o$. In general, a greater difference between the line and load impedances (a greater **mismatch**), will result in a greater percentage of the signal being reflected back toward the source.

Input Impedance

Input impedance is the impedance (ratio of total voltage to total current) seen at the input to a system.

Consider the impedance seen at the input port of this system:



where the variables are defined by:

$$V_{in} = V(-\ell) = total\ voltage\ seen\ at\ the\ port$$

$$I_{in} = I(-\ell) = total\ current\ seen\ at\ the\ port$$

$$Z_{in} = Z(-\ell) = input\ impedance\ of\ the\ port = {}^{V_{in}}/_{I_{in}}$$

Input Impedance

Input impedance is the impedance (ratio of total voltage to total current) seen at the input to a system.

$$\Gamma = \frac{V_o^-}{V_o^+} = \frac{Z_L - Z_o}{Z_L + Z_o}$$

Note that we also have the following relationships:

$$V_{in} = V(-\ell) = V_\circ^+ e^{\gamma\ell} + V_\circ^- e^{-\gamma\ell} = V_\circ^+\left(e^{-\gamma\ell} + \Gamma e^{+\gamma\ell}\right)$$

$$I_{in} = I(-\ell) = \frac{V_\circ^+}{Z_\circ} e^{+\gamma\ell} - \frac{V_\circ^-}{Z_\circ} e^{-\gamma\ell} = \frac{V_\circ^+}{Z_\circ}\left(e^{-\gamma\ell} - \Gamma e^{+\gamma\ell}\right)$$

so that:

$$Z_{in} = \frac{Z_\circ\left(e^{+\gamma\ell} + \Gamma e^{-\gamma\ell}\right)}{\left(e^{+\gamma\ell} - \Gamma e^{-\gamma\ell}\right)} = \frac{Z_\circ\left(e^{+\gamma\ell} + \dfrac{Z_L - Z_o}{Z_L + Z_o} e^{-\gamma\ell}\right)}{\left(e^{+\gamma\ell} - \dfrac{Z_L - Z_o}{Z_L + Z_o} e^{-\gamma\ell}\right)}$$

which simplifies to:

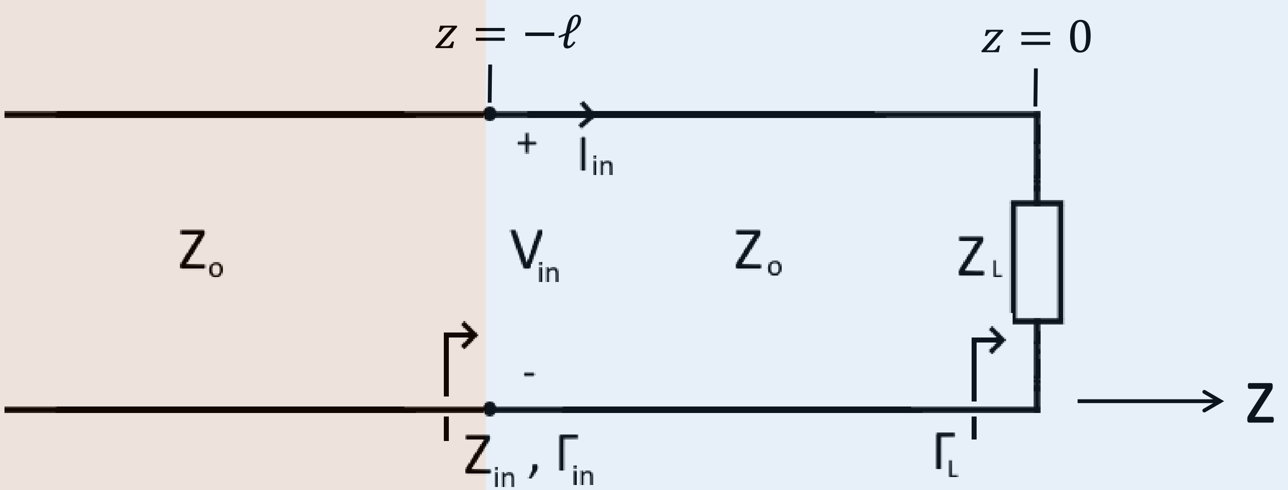
$$\boxed{Z_{in} = Z_\circ \frac{Z_L + jZ_\circ \tan(\beta\ell)}{Z_\circ + jZ_L \tan(\beta\ell)}}$$

This is the input impedance – the impedance seen at the input to a line with characteristic impedance $Z_\circ$, if the line is terminated with load impedance Z_L.

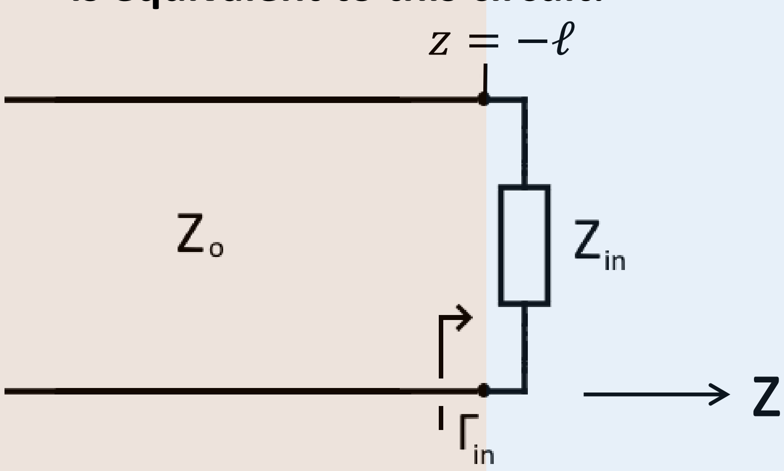
Input Impedance

Input impedance is the impedance (ratio of total voltage to total current) seen at the input to a system.

The significance of the input impedance is that this circuit:



is equivalent to this circuit:



Everything to the right of the input point may be replaced by Z_{in} without affecting the left side of the circuit.

It should also be noted that, for a lossless transmission line,

$$|\Gamma_L| = |\Gamma_{in}|$$

Input Impedance

Input impedance is the impedance (ratio of total voltage to total current) seen at the input to a system.

Let us look at two special cases:

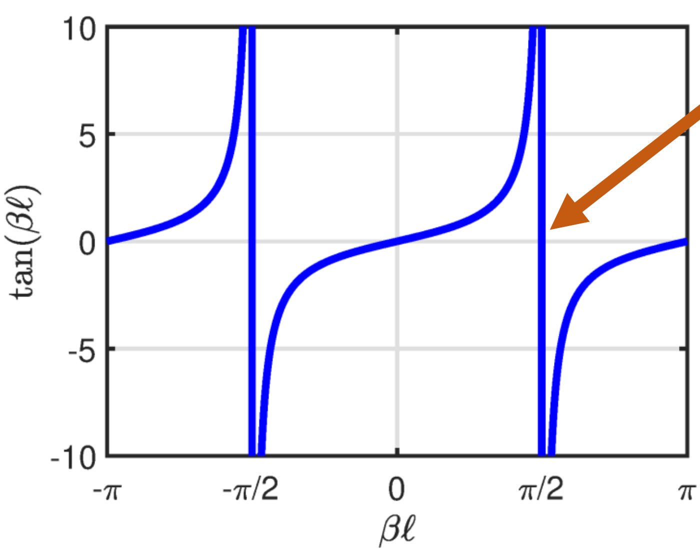
Recall

$$Z_{in} = Z_\circ \frac{Z_L + jZ_\circ \tan(\beta\ell)}{Z_\circ + jZ_L \tan(\beta\ell)}$$

Case 1: short circuit termination ($Z_L = 0$)

In this case,

$$\boxed{Z_{in,sc} = jZ_\circ \tan(\beta\ell) = jZ_\circ \tan\left(\frac{2\pi\ell}{\lambda}\right)}$$



Notes:

1) When $\ell = \frac{\lambda}{4}$, $\beta\ell = \frac{\pi}{2}$, and a short circuit ($Z_\circ = 0$) looks like an open circuit ($Z_{in} = \infty$)

2) The input impedance (Z_{in}) is inductive for $0 \le \ell \le \frac{\lambda}{4}$, and capacitive for $\frac{\lambda}{4} \le \ell \le \frac{\lambda}{4}$

3) The input impedance repeats every $\frac{\lambda}{2}$ $\left(Z_{in}(z) = Z_{in}\left(z + \frac{\lambda}{2}\right)\right)$

Input Impedance

Input impedance is the impedance (ratio of total voltage to total current) seen at the input to a system.

Let us look at two special cases:

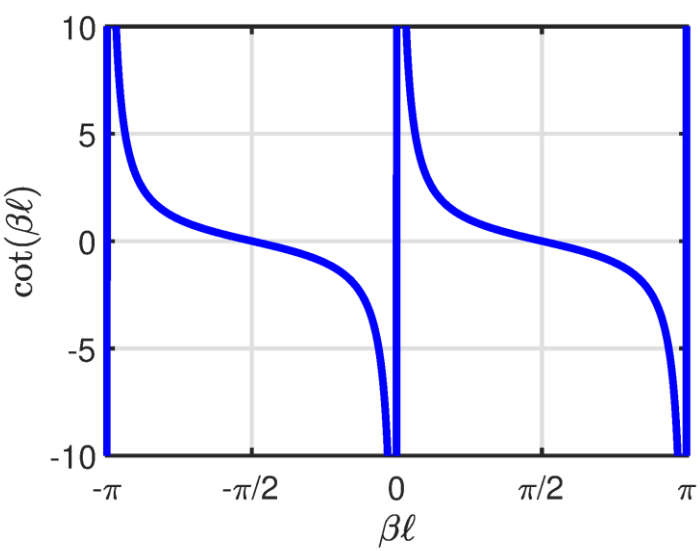
Recall

$$Z_{in} = Z_{\circ}\frac{Z_L + jZ_{\circ}\tan(\beta\ell)}{Z_{\circ} + jZ_L\tan(\beta\ell)}$$

Case 2: open circuit termination ($Z_L = \infty$)

In this case,

$$\boxed{Z_{in,oc} = -jZ_{\circ}\cot(\beta\ell) = -jZ_{\circ}\cot\left({}^{2\pi\ell}/_{\lambda}\right)}$$



Notes:

1) When $\ell = \frac{\lambda}{4}$, $\beta\ell = \frac{\pi}{2}$, and an open circuit ($\mathrm{Z}_{\circ} = \infty$) looks like an open circuit ($\mathrm{Z}_{in} = 0$)

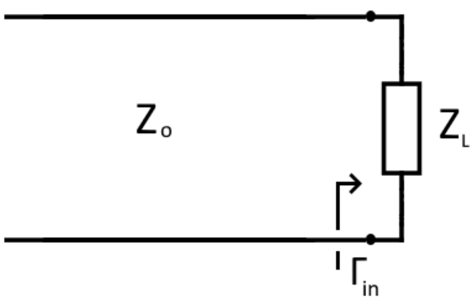
2) The input impedance (Z_{in}) is capacitive for $0 \leq \ell \leq \frac{\lambda}{4}$, and inductive for $\frac{\lambda}{4} \leq \ell \leq \frac{\lambda}{4}$

3) The input impedance repeats every $\frac{\lambda}{2}$ $\left(\mathrm{Z}_{in}(z) = Z_{in}\left(z + \frac{\lambda}{2}\right)\right)$

Quarter-wave Transformers

Power transfer to the load may be improved through addition of a "matching network" between the line and the load.

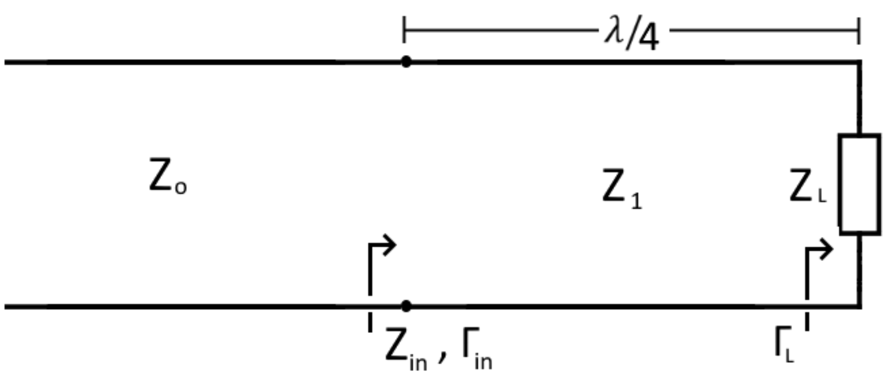
Consider this circuit: a load impedance of Z_L is being driven using a line with impedance $Z_\circ$.



In the most general case, $Z_L \neq Z_\circ$, and a portion of the signal reflects, according to:

$$\Gamma_{in} = \frac{Z_L - Z_\circ}{Z_L + Z_\circ}$$

To reduce the reflection, an engineer may choose to add a **matching network**. One such network is a **quarter-wave transformer**, which adds an extra quarter-wave-long section of transmission line before the load, like this:



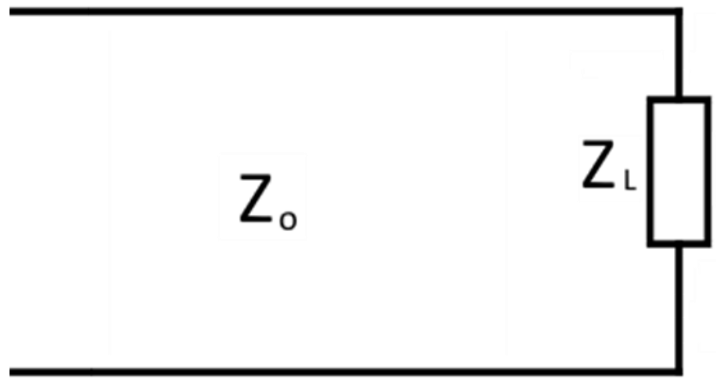
Here, if the added section has characteristic impedance $Z_1 = \sqrt{Z_L Z_\circ}$, then the input impedance at the port becomes $Z_{in} = Z_\circ$, which *eliminates the reflection* ($\Gamma_{\text{in}} = 0$)!

***Note: for this to work, Z_L must be purely real.

Impedance Discontinuities

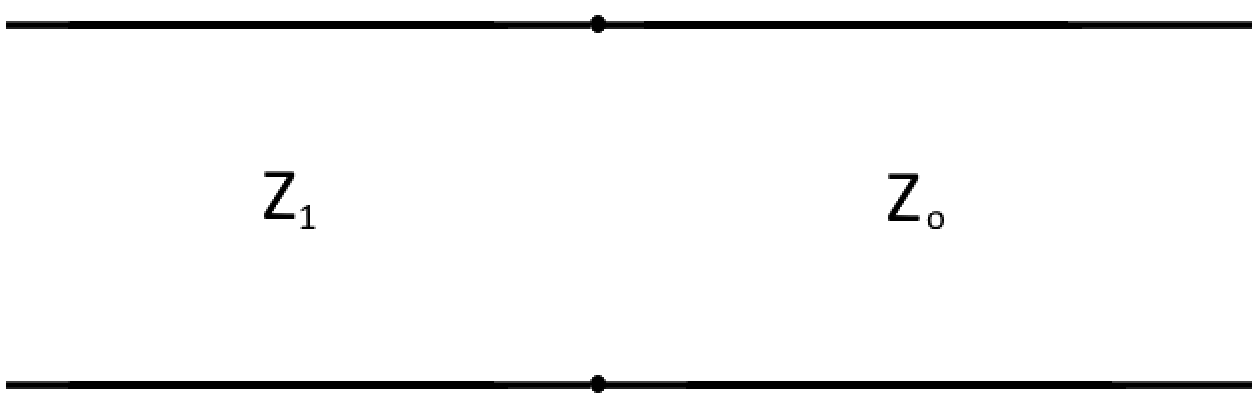
Reflections are caused by impedance discontinuities.

So far, we have discussed impedance discontinuities arising from a terminating load impedance that differs from the line impedance.



$Z_L \neq Z_o$

Reflections will also occur at line impedance discontinuities, as shown in the circuit below, where the line impedance value changes from Z_1 to Z_0.



$Z_1 \neq Z_o$

Impedance Discontinuities

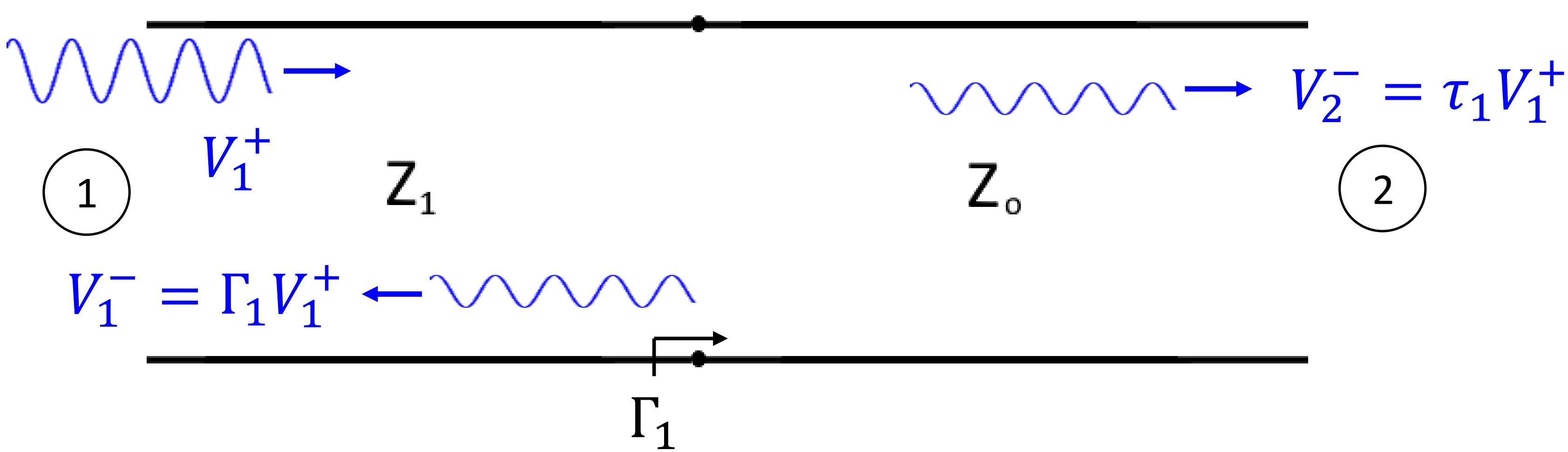
Reflections are caused by impedance discontinuities.

If a wave is input from port 1, at the left (V_1^+), it will experience an instantaneous reflection at the discontinuity. The reflected wave (V_1^-) will have a magnitude of $\Gamma_1 V_1^+$, where:

$$\Gamma_1 = \frac{Z_\circ - Z_1}{Z_\circ + Z_1}$$

A portion of the wave will also be transmitted into the $Z_\circ$ line, with voltage magnitude calculated using the **transmission coefficient** τ_1, as $\tau_1 V_1^+$, where:

$$\tau_1 = 1 + \Gamma_1$$



***Notice that the sum of the reflected and transmitted voltage magnitudes is not equal to the input voltage magnitude. This is because it is **power** that is conserved, **not voltage.**

Impedance Discontinuities

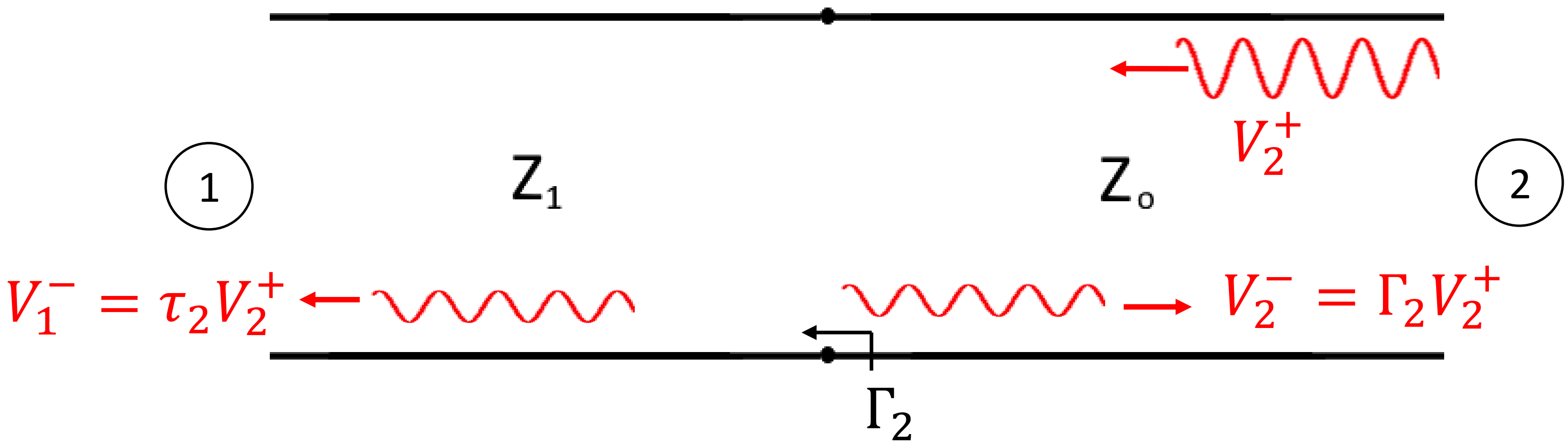
Reflections are caused by impedance discontinuities.

Similarly, if a wave is input from the right (V_2^+), it will experience an instantaneous reflection at the discontinuity. The reflected wave (V_2^-) will have a magnitude of $\Gamma_2 V_2^+$, where:

$$\Gamma_2 = \frac{Z_1 - Z_\circ}{Z_1 + Z_\circ}$$

A portion of the wave will also be transmitted into the Z_1 line, with voltage magnitude calculated using the **transmission coefficient** τ_2, as $\tau_2 V_2^+$, where:

$$\tau_2 = 1 + \Gamma_2$$



***Notice that the sum of the reflected and transmitted voltage magnitudes is not equal to the input voltage magnitude. This is because it is **power** that is conserved, **not voltage.**

Notational Convention

In naming these variables, we are using the following conventions:

In naming these variables, we are using the following conventions:

V_2^+

V_2^-

The superscript on the voltage terms indicates the direction that the voltage wave is traveling.

+ indicates voltage *entering* the system (input voltage)

- indicates voltage *exiting* the system (output voltage)

V_1^+

V_2^+

The subscript on the input voltage terms indicates the entry location of the wave.

1 indicates entry at port 1

2 indicates entry at port 2

V_1^-

V_2^-

The subscript on the output voltage terms indicates the exit location of the wave.

1 indicates exit at port 1

2 indicates exit at port 2

Γ_1

τ_2

The subscript on the coefficients indicates their perspective.

1 corresponds to a coefficient looking in from port 1

2 corresponds to a coefficient looking in from port 2

Note that **perspective of a coefficient matters!** Case in point:

$$\boxed{\Gamma_1 = -\Gamma_2}$$

S-parameters

S-parameters are a way of externally characterizing a system by analyzing the transmissions and reflections of voltage signals at its ports.

Consider the following 2-port network:

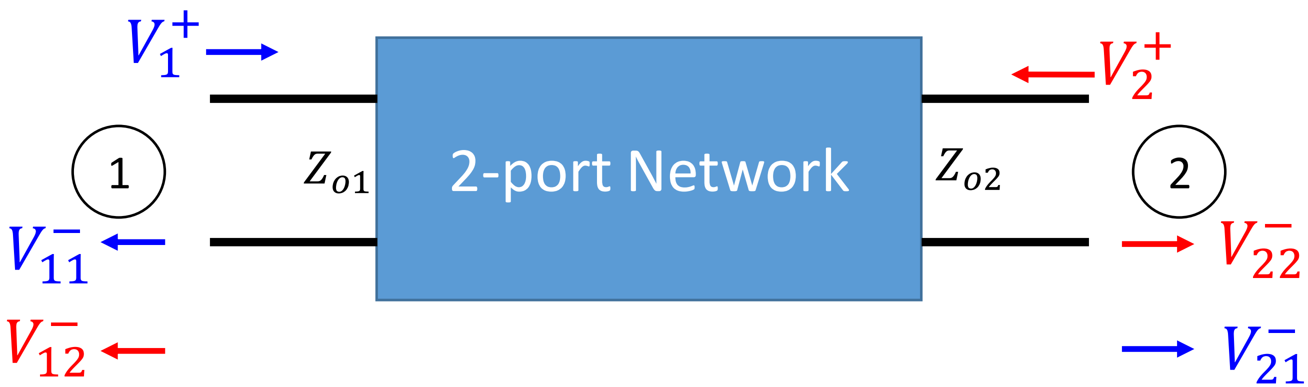
Notes:

- A "port" is a location where energy (V, I) may enter and exit the system.
- An N-port network has N ports (in this case, two).
- Each port has its own characteristic impedance.
- The network may be driven at either of the two ports.
- Each driven signal may experience reflection (back out its entry port) and/or transmission to the other port.

S-parameters

S-parameters are a way of externally characterizing a system by analyzing the transmissions and reflections of voltage signals at its ports.

This is the complete set of signal path possibilities, including input at either port.



Here, we must include an extra specifier in our nomenclature. Note that the total voltage output at port 1 includes the reflected component of V_1^+, which we have named V_{11}^-, and also the transmitted component of V_2^+, which we have named V_{12}^-. Similar conventions were followed for port 2. Thus,

V_{11}^-

V_{12}^-

For variables having two subscripts, the first subscript indicates the **exit port** of the energy (in this case, port 1) and the second subscript indicates the **entry port** of the energy.

S-parameters

S-parameters are a way of externally characterizing a system by analyzing the transmissions and reflections of voltage signals at its ports.

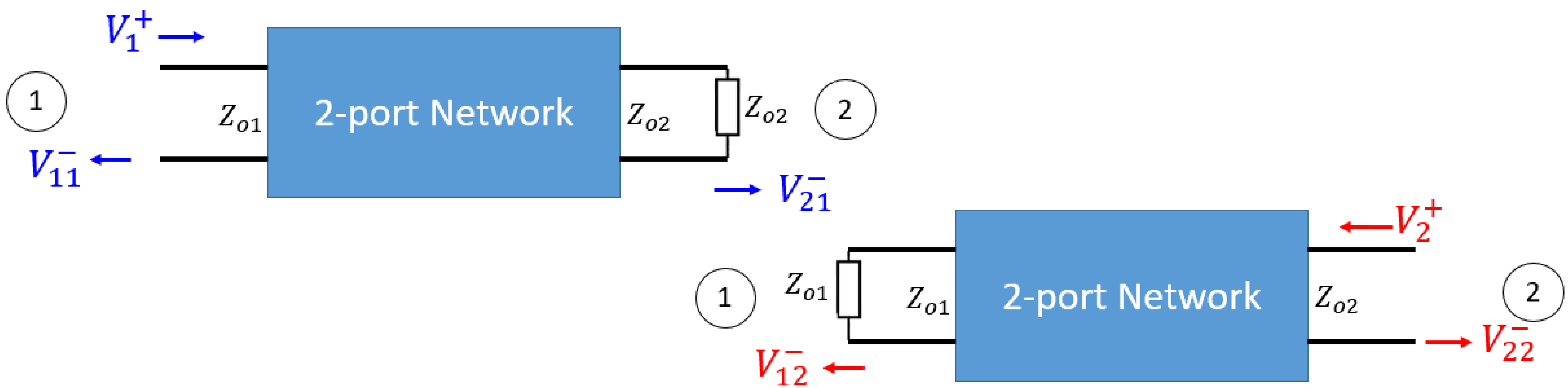
In order to analyze the system, we must be able to parse out how much of the output voltage is due to reflection, and how much is due to transmission. To do this, we conduct N measurements. For each measurement, we will drive at a single port, and match-terminate (load with its own characteristic impedance) all other ports.

Measurement 1:

Measurement 2:

S-parameters

S-parameters are a way of externally characterizing a system by analyzing the transmissions and reflections of voltage signals at its ports.



Please note the following:

- For accurate measurements, the input at all non-driven ports *must* be zero.
- If the non-driven port were not match-terminated, the signal exiting that port would be partially reflected at the load, and this reflected signal would *re-enter the system*, appearing as an input at that port.
- Match-termination ensures that *all* the signal exiting the passive port is absorbed by the load, and *none reflects back into the circuit*.

S-parameters

S-parameters are a way of externally characterizing a system by analyzing the transmissions and reflections of voltage signals at its ports.

For an N-port network, the S-parameters will be represented by an $N \times N$ matrix. As an example, a 2-port network will have the following S-parameters, or scattering-matrix:

$$[S] = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$$

These parameters are interpreted as follows:

S_{11} = reflection at port 1 $= V_{11}^{-}/V_{1}^{+}$

S_{12} = transmission **to** port 1 **from** port 2 $= V_{12}^{-}/V_{2}^{+}$

S_{21} = transmission **to** port 2 **from** port 1 $= V_{21}^{-}/V_{1}^{+}$

S_{22} = reflection at port 2 $= V_{22}^{-}/V_{2}^{+}$

S-parameters

S-parameters are a way of externally characterizing a system by analyzing the transmissions and reflections of voltage signals at its ports.

In general,

$$S_{ij} = \frac{\sqrt{Z_{\circ j}}}{\sqrt{Z_{\circ i}}} \frac{Voltage\ coming\ out\ at\ port\ i}{Voltage\ going\ in\ at\ port\ j} \Bigg|_{\text{input at all other ports}=0}$$

Or,

$$S_{ij} = \frac{\sqrt{Z_{\circ j}}}{\sqrt{Z_{\circ i}}} \frac{V_i^-}{V_j^+} \Bigg|_{\mathrm{V_k^+=0,k\neq j}}$$

Note that, if the ports have the same characteristic impedance ($Z_{\circ j} = Z_{oi}$ for all values of i and j), this simplifies to:

$$S_{ij} = \frac{V_i^-}{V_j^+} \Bigg|_{\mathrm{V_k^+=0,k\neq j}}$$

Moving forward, **we will assume that all systems have equal characteristic impedance at all ports** ($Z_{\circ j} = Z_{oi}$ for all values of i and j). In reality, this is usually (but not always) the case. If you ever need to deal with mismatched port impedances, scale all voltages at each port by the characteristic impedance at that port.

S-parameters

S-parameters are a way of externally characterizing a system by analyzing the transmissions and reflections of voltage signals at its ports.

$$S_{ij} = \frac{V_i^-}{V_j^+}\bigg|_{V_k=0, k\neq j}$$

Applying this to a 2-port network returns the following:

$$S_{11} = \frac{V_1^-}{V_1^+}\bigg|_{V_2^+=0} \qquad S_{12} = \frac{V_1^-}{V_2^+}\bigg|_{V_1^+=0}$$

$$S_{21} = \frac{V_2^-}{V_1^+}\bigg|_{V_2^+=0} \qquad S_{22} = \frac{V_2^-}{V_2^+}\bigg|_{V_1^+=0}$$

So that the **total** voltage coming out each port is given by:

$$V_1^- = V_1^+ S_{11} + V_2^+ S_{12}$$
$$V_2^- = V_1^+ S_{21} + V_2^+ S_{22}$$

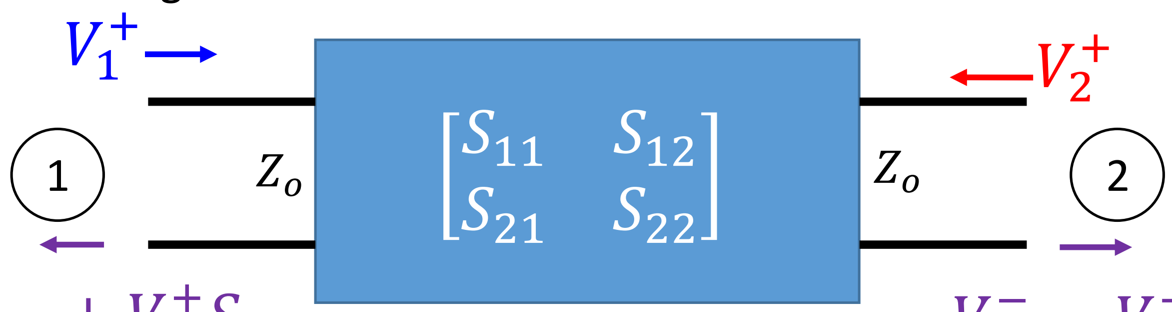
or,

$$\begin{bmatrix} V_1^- \\ V_2^- \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} V_1^+ \\ V_2^+ \end{bmatrix}$$

S-parameters: Losslessness

S-parameters are a way of externally characterizing a system by analyzing the transmissions and reflections of voltage signals at its ports.

Consider the following:



$V_1^- = V_1^+ S_{11} + V_2^+ S_{12}$

$V_2^- = V_1^+ S_{21} + V_2^+ S_{22}$

If this network is <u>*lossless:*</u>

$$|V_1^+|^2 = |S_{21}V_1^+|^2 + |S_{11}V_1^+|^2$$
$$|V_2^+|^2 = |S_{12}V_2^+|^2 + |S_{22}V_2^+|^2$$

➡

$$\boxed{\begin{aligned} 1 &= |S_{21}|^2 + |S_{11}|^2 \\ 1 &= |S_{12}|^2 + |S_{22}|^2 \end{aligned}}$$

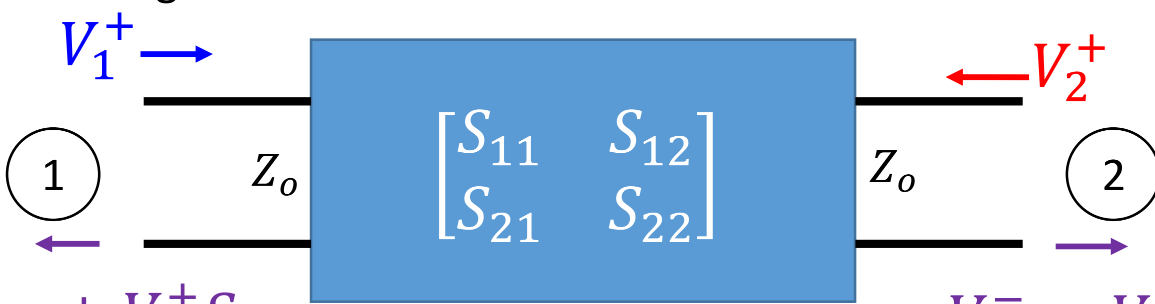
➡ $\boxed{\sum_{i=1}^{N} |S_{ij}| = 1, \text{ for all values of j.}}$

In a lossless system, the sum of all squared magnitudes of the elements in each column of the S-matrix is equal to one.

S-parameters: Reciprocity

S-parameters are a way of externally characterizing a system by analyzing the transmissions and reflections of voltage signals at its ports.

Consider the following:



$$V_1^- = V_1^+ S_{11} + V_2^+ S_{12}$$

$$V_2^- = V_1^+ S_{21} + V_2^+ S_{22}$$

If this network is *reciprocal:*

$$\boxed{[S] = [S]^T} \xrightarrow{\text{for a 2-port network,}} [S] = \begin{bmatrix} S_{11} & S_{21} \\ S_{12} & S_{22} \end{bmatrix}, \; S_{21} = S_{12}$$

In a reciprocal system, the S-matrix is symmetric about the main diagonal.

Return Loss and Insertion Loss

Let us revisit return loss, and introduce insertion loss…

We have previously defined return loss for a single-port network as:

$$\boxed{\mathrm{RL} = -20\log_{10}|\Gamma| \quad \text{, dB}}$$

For an N-port network, we will now define **return loss** in the context of S-parameters:

$$\boxed{\mathrm{RL}_n = -20\log_{10}|\mathrm{S}_{nn}| \text{ , dB}}$$

And we will also introduce **insertion loss**, having to do with the transmission between ports I and j

$$\boxed{\mathrm{IL}_{ij} = -20\log_{10}\left|\mathrm{S}_{ij}\right| \ ,i \neq j \text{ , dB}}$$

As an example, for a 2-port network, we will have the following:

$$\mathrm{RL}_1 = -20\log_{10}|\mathrm{S}_{11}| \qquad \mathrm{IL}_{12} = -20\log_{10}|\mathrm{S}_{12}|$$

$$\mathrm{RL}_2 = -20\log_{10}|\mathrm{S}_{22}| \qquad \mathrm{IL}_{21} = -20\log_{10}|\mathrm{S}_{21}|$$