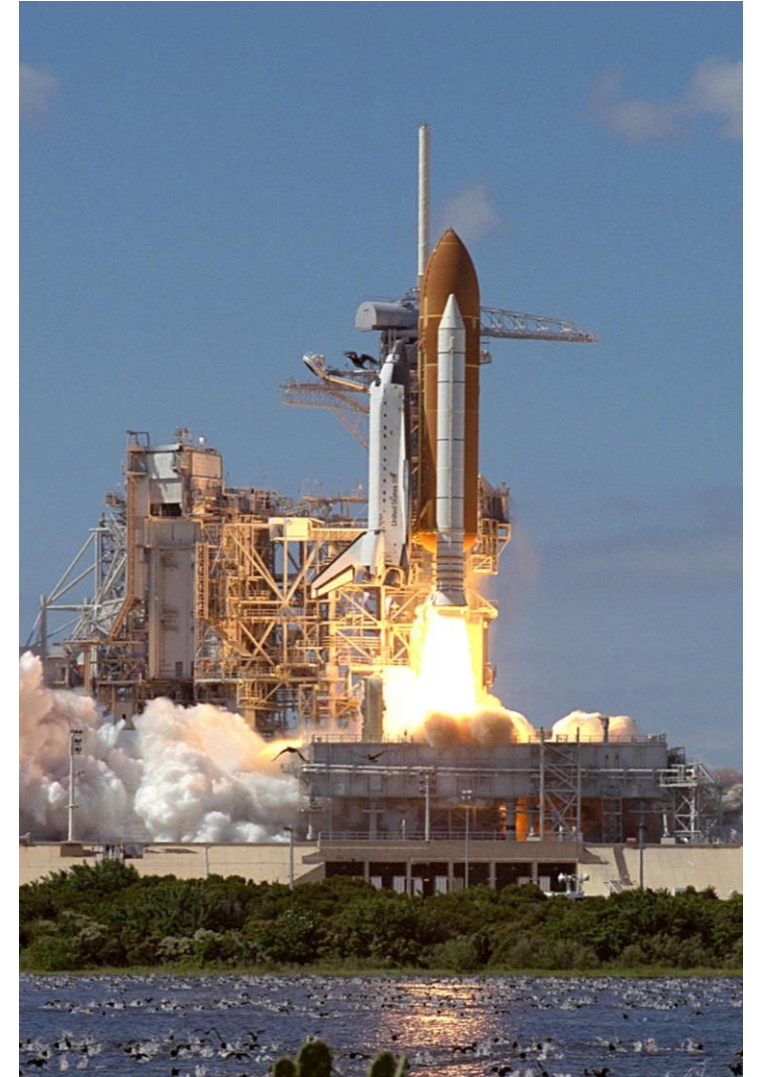# Conservation of Energy Equation

Governing Equations of Fluid Dynamics – Lesson 5

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Introduction

- We now turn our attention to the conservation of energy equation, which is a restatement of the First Law of Thermodynamics.

- For problems involving heat transfer and compressible flows, this equation will permit us to obtain the temperature of the fluid, thereby adding one more unknown to our model along with density, pressure and the three velocity components.

- If the density is a function of temperature, we will also need an equation relating density to temperature (and pressure). This comes in the form of a thermodynamic equation of state.

- However, if compressibility and heat transfer are not considered, we can assume the fluid is isothermal (constant temperature), and thus the conservation of energy equation will not be required.

Conservation of Energy

- The First Law of Thermodynamics can be stated as follows:

  💡 *The total energy change in system equals the difference between the heat transferred to the system and the work done by the system on its surroundings.*

- For our Lagrangian fluid parcel considered earlier, this can be stated as:

$$\frac{dE}{dt} = \dot{Q} - \dot{W}$$

heat transfer   work

Note: If the work is assumed to be done on the system by its surroundings, the equation turns into:

$$\frac{dE}{dt} = \dot{Q} + \dot{W}$$

- In order to proceed further, we need to define the processes of heat transfer and work for the fluid parcel (the right-hand side of the above equation) and address what is meant by the total energy E.

Total Energy

- The energy per unit mass contained in a system is comprised of three parts: internal, kinetic and potential.

$$e_t \equiv e(total) = e(internal) + e(kinetic) + e(potential)$$

  - The internal energy per unit mass of the fluid is simply denoted here as e. This is a function of the fluid temperature.

$$e(internal) = e(T)$$

  - The kinetic energy per unit mass is expressed through the fluid velocity as:

$$e(kinetic) = \frac{1}{2}V^2$$

  - The potential energy per unit mass can come from many sources but typically includes just the gravitational potential energy for a fluid at a local elevation (height) of z:

$$e(potential) = gz$$

- We can therefore write the total energy E in the Lagrangian parcel as an integral over the parcel volume:

$$E = \iiint_\Omega \rho\, e_t\, d\Omega = \iiint_\Omega \rho \left[e(T) + \frac{1}{2}V^2 + gz \right] d\Omega$$

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Equation of State

- To close the energy equation, we need two things: an equation of state and thermodynamic relations which relate the energy variables to basic properties such as temperature and pressure.

- We can express the equation of state for a homogeneous substance as a general density function of pressure and temperature as follows:

$$\rho = \rho(p, T)$$

  o The most common example of an equation of state is the Ideal Gas Law

$$\rho = \frac{p}{RT}$$

    Where R = gas constant specific to the fluid

  o The Ideal Gas Law applies to most gases at standard atmospheric pressure and temperatures (e. g., air, carbon dioxide, water vapor).

- If we have a non-ideal gas behavior (for example, a pure vapor near saturation or the critical point), the equation of state can be very complex. This is often referred to as real gas behavior. Real gas behavior will not be considered further in this course but can be introduced if needed.

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Thermodynamic Relations

- For compressible flow, there are many thermodynamic relations which permit conversion of energy variables into various forms. We will not be considering this topic in any depth here but will simply state some useful relations applicable to ideal gases (which comprise many of the problems we are interested in).

Internal energy (e):

$$e = C_v T$$

Enthalpy (h):

$$h = e + \frac{p}{\rho} = C_p T$$

Specific heat ratio (γ):

$$\gamma = \frac{C_p}{C_v} \qquad C_v = \frac{R}{\gamma - 1} \qquad C_p = \frac{\gamma R}{\gamma - 1}$$

Total internal energy (e_t):

$$e_t = e + \frac{V^2}{2}$$

Total enthalpy (h_t):

$$h_t(T) = h + \frac{V^2}{2}$$

Ideal Gas Law relations:

$$p = (\gamma - 1)\rho e \qquad T = \frac{(\gamma - 1)e}{R}$$

Energy Source Terms

- The time rate of change of the total energy E can be impacted by other processes not represented in the internal, kinetic, and potential energies. We can account for these sources by adding a generation source term of energy ($\dot{S}_g$) to our original First Law formulation:

$$\frac{dE}{dt} = \dot{Q} - \dot{W} + \dot{S}_g$$

- Typical sources can include radiation effects and thermal heating from electrical current flow among many other physical effects. Further consideration of these effects are beyond the scope of this course.

Heat Transfer

- The primary mechanism for heat transfer $\dot{Q}$ to the fluid parcel is thermal conduction.

- Since the heat is transferred to/from the system across the boundary surface, we can represent it as a surface integral of the heat flux $\dot{q}''$ (W/m²)

$$\dot{Q} = -\oiint_A \dot{q}'' \, dA$$

- For conduction heat transfer, the heat flux is related to the temperature gradient via Fourier's Law:

where k is the thermal conductivity of the fluid $(W/m - K)$.

$$\dot{q}'' = -k \, \nabla T$$

Work Done on the Surroundings

- In thermodynamics, there are two primary mechanisms of work that is done by a system on its surroundings: Pressure Work and Viscous Shear Work.

  - Pressure Work – This is the work done by the pressure acting normal to the system boundary. If the velocity on the boundary is non-zero, then this contribution is given by:

$$\dot{W}(pressure) = \oiint_A p(\vec{V} \cdot \hat{n})\, dA$$

  - Viscous Work – This is the work done by the action of the fluid viscous shear and is also known as the viscous dissipation term. As with the pressure work, this can be expressed as a surface integral involving the shear stress tensor $\bar{\bar{\tau}}$ as follows:

$$\dot{W}(viscous) = \oiint_A (\bar{\bar{\tau}} \cdot \vec{V}) \cdot \hat{n}\, dA$$

Eulerian Form [1]

- Let us now develop the energy equation using the Reynolds Transport Theorem:

$$\frac{dE}{dt} = \frac{d}{dt}\iiint_\Omega \rho e_t d\Omega = \iiint_\Omega \frac{\partial(\rho e_t)}{\partial t}\, d\Omega + \oiint_A e_t(\rho \vec{V}\cdot\hat{n})dA = \dot{Q} - \dot{W} + \dot{S}_g$$

- Inserting the terms for heat transfer and work shown previously yields:

$$\iiint_\Omega \frac{\partial(\rho e_t)}{\partial t}\, d\Omega + \oiint_A e_t(\rho \vec{V}\cdot\hat{n})dA = -\oiint_A \dot{q}''\, dA - \oiint_A p(\vec{V}\cdot\hat{n})\, dA + \oiint_A (\bar{\bar{\tau}}\cdot\vec{V})\cdot\hat{n}\, dA + \iiint_\Omega \dot{S}_g\, d\Omega$$

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Eulerian Form [2]

- We can place the pressure work term inside the convection term and further introduce Fourier's Law and simplify the right-hand side to give:

$$\iiint\limits_{\Omega} \frac{\partial(\rho e_t)}{\partial t}\, d\Omega + \oiint\limits_{A}\left(e_t + \frac{p}{\rho}\right)(\rho\vec{V}\cdot\hat{n})dA = \oiint\limits_{A}\left[k\,\nabla T + (\bar{\bar{\tau}}\cdot\vec{V})\right]\cdot\hat{n}\,dA + \iiint\limits_{\Omega}\dot{S}_g\,d\Omega$$

- We now apply the Divergence Theorem to the surface integrals in the above:

$$\iiint\limits_{\Omega} \frac{\partial(\rho e_t)}{\partial t}\, d\Omega + \iiint\limits_{\Omega}\nabla\cdot\left[\rho\vec{V}\left(e_t + \frac{p}{\rho}\right)\right]d\Omega = \iiint\limits_{\Omega}\nabla\cdot\left[k\,\nabla T + (\bar{\bar{\tau}}\cdot\vec{V})\right]d\Omega + \iiint\limits_{\Omega}\dot{S}_g\,d\Omega$$

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Conservation of Energy Equation – Final Form

- The control volume is arbitrary, hence the sum of all the integrands must be zero to satisfy the equilibrium. Finally, we obtain the differential form of the conservation of energy equation:

$$\frac{\partial(\rho e_t)}{\partial t} + \nabla \cdot \left[\vec{V}(\rho e_t + p)\right] = \nabla \cdot \left[k\,\nabla T + (\bar{\bar{\tau}} \cdot \vec{V})\right] + \dot{S}_g$$

- Note that for low speed flows, the total energy per unit mass can be related directly to temperature and thus we can convert the energy equation into an equation for temperature.

- As it stands, we will need to supplement the energy equation with an equation of state and thermodynamic relations in order to form a closed set of equations.

- The Cartesian form of the equation can be easily derived from its vector form, as was done for the momentum equations in previous lessons. Here we are not going to write its full Cartesian form and instead introduce its variants when discussing selected classes of heat transfer.

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Comments on the Energy Equation

- For single phase compressible fluids with no advanced physical effects (such as combustion or radiation), our final set of governing equations for fluid mechanics are conservation of mass, conservation of momentum and conservation of energy.

- This set of equations, along with an equation of state and suitable initial/boundary conditions, allows us to model the entire flow field and solve for the unknown flow variables at every point in space (and for unsteady problems at all times).

- Of course, due to the complexity of these equations, we need to impose suitable modeling assumptions in order to develop a tractable model.

Summary

- We have considered the conservation of energy equation in this lesson.

- For general compressible flows, we require both the conservation of energy equations and an equation of state to close the system.

- For low-speed flows, conservation of energy can be simplified to an equation for temperature alone.