Method of Characteristics

Internal Compressible Flows – Lesson 6

Intro

- In this lesson we will discuss the method of characteristics for solving the governing equations of supersonic steady inviscid and irrotational flows, and illustrate an application of this method in a design of a divergent section of a supersonic nozzle.

- Combining the Continuity equation and Euler's equation, under the assumption of two-dimensional irrotational flow, we can derive the velocity potential equation:

$$\left[1-\frac{\Phi_x^2}{a^2}\right]\Phi_{xx}+\left[1-\frac{\Phi_y^2}{a^2}\right]\Phi_{yy}-\frac{2\Phi_x\Phi_y}{a^2}\Phi_{xy}=0$$

- In a supersonic flow, this equation is of hyperbolic type and can be solved by the method of characteristics.

- It is not our goal here to go over the entire mathematical theory of hyperbolic equations. Instead, we will simply refer to PDE textbooks for the main results needed for this lesson.

Main Postulates

- A PDE is hyperbolic if the coefficients of its highest-order derivatives satisfy a certain relation. For the velocity potential equation, this relation is:

$$\left(\Phi_x^2 + \Phi_y^2\right)/a^2 > 1$$

- A hyperbolic equation is characterized by the existence of certain directions (or lines) in the $x - y$ plane called characteristics. For the velocity potential equation, characteristics are the Mach lines.

- The normal derivatives of the velocity components (2nd derivatives of the velocity potential) are indeterminate on a characteristic and can even be discontinuous in some cases, but the velocity itself (or 1st derivatives of the velocity potential) is continuous.

- Since the normal velocity derivative is allowed to be discontinuous on a characteristic, different flows can be patched together at characteristic lines.

- On characteristics, the velocity potential satisfies a certain relation called the compatibility equation.

Method of Characteristics

- Let's go back to the 2D velocity potential equation.
- Since the velocity potential and its derivatives are functions of x and y, we have:

$$d\left(\frac{\partial\Phi}{\partial x}\right) = du = \frac{\partial^2\Phi}{\partial x^2}dx + \frac{\partial^2\Phi}{\partial x\,\partial y}dy \qquad d\left(\frac{\partial\Phi}{\partial y}\right) = dv = \frac{\partial^2\Phi}{\partial y^2}dy + \frac{\partial^2\Phi}{\partial x\,\partial y}dx$$

- As these equations represent three equations with three unknowns, using Cramer's rule we get:

$$\frac{\partial^2\Phi}{\partial x\partial y} = \begin{vmatrix} 1-\frac{u^2}{a^2} & 0 & 1-\frac{v^2}{a^2} \\ dx & du & 0 \\ 0 & dv & dy \end{vmatrix} \begin{vmatrix} 1-\frac{u^2}{a^2} & -\frac{2uv}{a} & 1-\frac{v^2}{a^2} \\ dx & dy & 0 \\ 0 & dx & dy \end{vmatrix}^{-1} = \frac{N}{D}$$

Characteristic Lines

- If the denominator D is chosen such that $D = 0$, then the numerator N must also be zero $(N = 0)$, as we know that $\frac{\partial^2\phi}{\partial x \partial y}$ has a specific defined value at every point in the flow.

- Thus, we can say that there is some direction at every point (A) along which $\frac{\partial^2\phi}{\partial x \partial y}$ is indeterminate which is the characteristic line. The precise direction of these lines can be calculated as follows:

- Consider the point A in the flow field and set the denominator D to zero. Expanding the determinant D and setting it to zero, we get:

$$\left[1 - \frac{u^2}{a^2}\right]\left(\frac{dy}{dx}\right)^2_{char} + \left[1 - \frac{v^2}{a^2}\right] + \frac{2uv}{a^2}\left(\frac{dy}{dx}\right)^2_{char} = 0$$

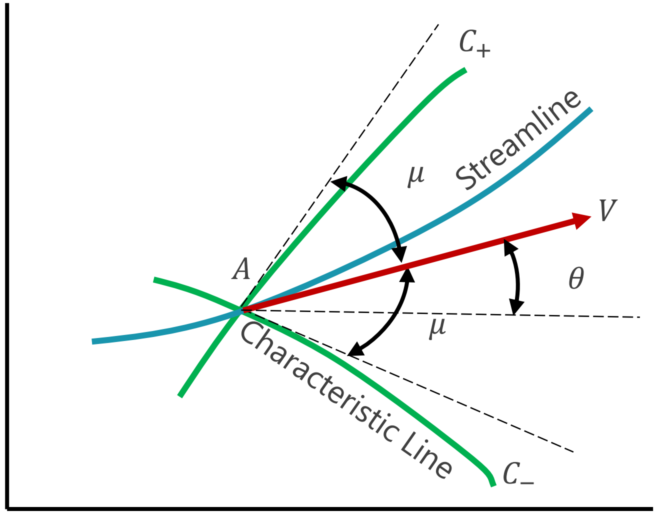
- Here $\left(\frac{dy}{dx}\right)_{char}$ represents the slope of the characteristic lines and is given by the roots of this quadratic equation:

$$\left(\frac{dy}{dx}\right)_{char} = \frac{\left(-\frac{uv}{a^2} \pm \sqrt{\frac{u^2+v^2}{a^2} - 1}\right)}{1 - \frac{u^2}{a^2}}$$

Characteristic Lines (cont.)

- Since $u = V cos\ \theta$ and $v = V sin\ \theta$, and the local Mach angle μ is given by $\mu = \sin^{-1} 1/M$, the slope becomes:

$$\left(\frac{dy}{dx}\right)_{char} = \left(-\frac{\cos\theta\sin\theta}{\sin^2\mu} \pm \sqrt{\frac{\cos^2\theta + \sin^2\theta}{\sin^2\mu} - 1}\right)\left(1 - \frac{cos^2\theta}{\sin^2\mu}\right)^{-1}$$



- After some algebraic and trigonometric manipulation, we get:

$$\left(\frac{dy}{dx}\right)_{char} = \tan(\theta \mp \mu)$$

- This equation states that the two characteristic lines running through the point A have slopes equal to $\tan(\theta - \mu)$ shown by C_- and $\tan(\theta + \mu)$ depicted by C_+.

- The characteristic lines through the point A are simply the left and right running Mach waves through the point, i.e., the characteristic lines are Mach lines.

- Note that the characteristic lines are curved in space because the local Mach angle depends on the local Mach number which is a function of both x and y. Moreover the local streamline direction θ also varies throughout the flow.

Compatibility Equations

- Along the characteristic lines, the governing partial differential equation describing the flow reduce to ordinary differential equations known as compatibility equations.
- These can be found by setting the numerator determinant to zero, $N = 0$, giving:

$$\frac{dv}{du} = -\frac{\left(1 - \frac{u^2}{a^2}\right)}{1 - \frac{v^2}{a^2}}\left(\frac{dy}{dx}\right)_{char}$$

- Substituting here the slope of the characteristic lines, $u = V\cos\theta$ and $v = V\sin\theta$ and after some algebraic manipulations we get the following ODEs:

$$d\theta = -\sqrt{M^2 - 1}\frac{dV}{V} \qquad \text{along } C_-$$

$$d\theta = \sqrt{M^2 - 1}\frac{dV}{V} \qquad \text{along } C_+$$

Compatibility Equations (cont.)

- These equations can be integrated to obtain a result in terms of the Prandtl-Meyer function as shown:

$$\theta + \nu(M) = \text{const} = K_- \qquad \text{along } C_-$$

$$\theta - \nu(M) = \text{const} = K_+ \qquad \text{along } C_+$$

Here K_- and K_+ are different constants along different C_- and C_+ characteristics.

- The above compatibility equations are now reduced to algebraic equations. In general inviscid supersonic steady flow, the compatibility equations are ODEs.
- In the case of a 2D irrotational flow they further reduce to algebraic equations.
- The equations can be combined to obtain simple expressions to calculate θ and ν.

$$\theta = \frac{1}{2}\left[(K_-) + (K_+))\right]$$

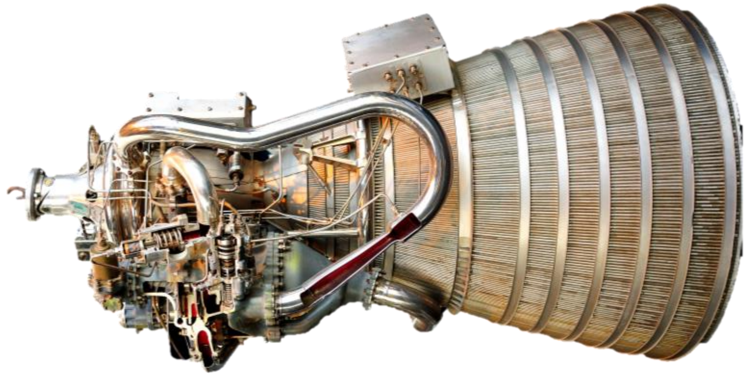
$$\nu = \frac{1}{2}\left[(K_-) - (K_+))\right]$$

- Next, we discuss how we can use these results to calculate the supersonic flow inside a nozzle and determine a proper wall contour so that no shock waves appear inside the nozzle.

Supersonic Nozzle Design

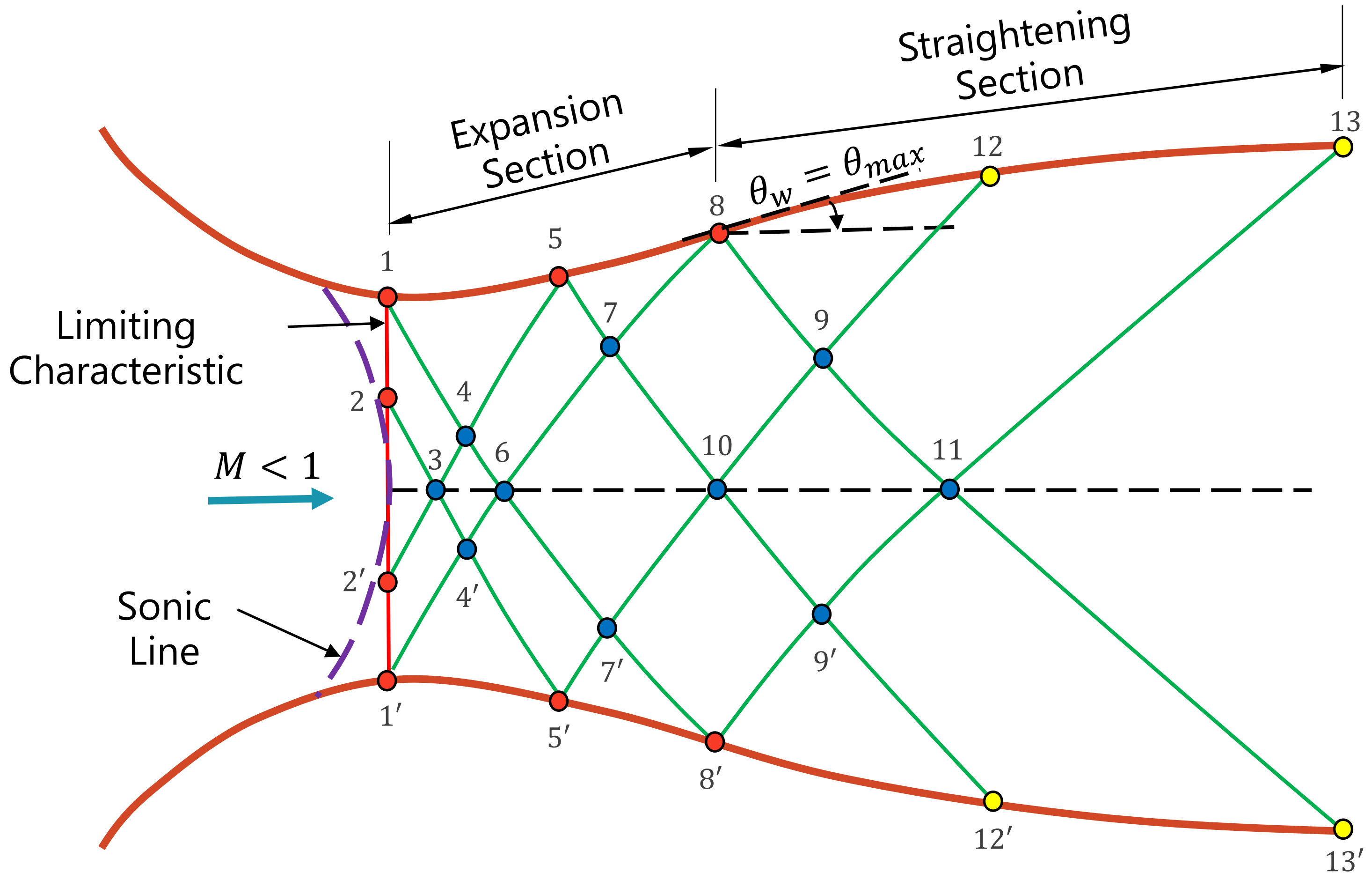
- Design problem:

  Design the wall contour for a converging-diverging nozzle to allow shock-free isentropic expansion of a gas from rest to a given supersonic Mach number at the exit.



- For the convergent section, there is no particular contour which gives better results than the others. Thus the design of this part of the nozzle is usually based on experience and industry best practices.
- Let's consider a 2D flow for simplicity, as shown on the next slide. The sonic line is located at the throat.
- The limiting characteristic is such that any characteristic line originating downstream of this line does not intersect the sonic line.
- Based on the calculations of the convergent profile, we know the flow properties in the throat region and thus we can use the limiting characteristic as the initial data line.
- The diverging section is divided into two regions: the expansion section and straightening section.

Supersonic Nozzle Design (cont.)



Supersonic nozzle profile design using method of characteristics

Supersonic Nozzle Design (cont.)

- If we define θ_w as the angle between a tangent to the wall and the horizontal, then the section where θ_w is increasing is called the expansion section. This section ends at the point of $\theta_w = \theta_{max}$.

- Downstream from this point is the straightening section where θ_w decreases to zero at the nozzle exit.

- The shape of the expansion section is arbitrary and is usually in the form of a circular arc of large radius.

- Therefore, for our analysis we know the flow properties along the limiting characteristic line and the slope at points 1, 5 and 8. Now we simply need to apply the method of characteristics to design the contour of the straightening section, i.e., from points 8 through 13. Points below the axis are simply the reflections of the points above the axis.

- It should be noted that the characteristic mesh sketched here is very coarse; in actual calculations the mesh should be much finer.

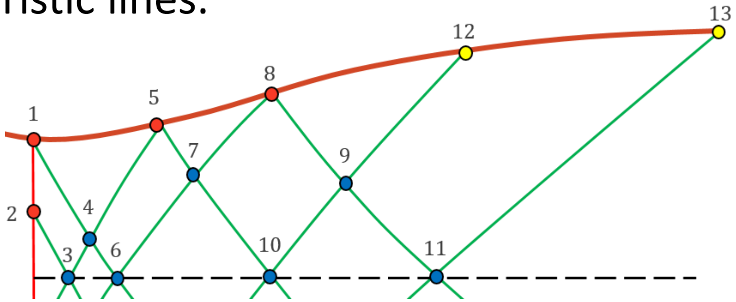
- The method of characteristics is an exact solution of inviscid, nonlinear supersonic flow. However, in practice, there are numerical errors associated with the finite grid: the approximation of the characteristics mesh by straight-line segments between grid points is one such example.

- In principle, the method of characteristics is truly exact only in the limit of an infinite number of characteristic lines.

Internal Points

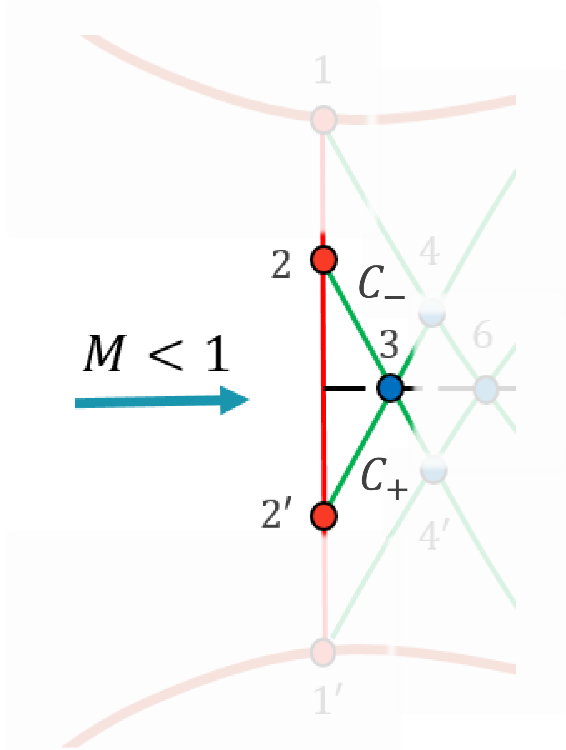
- The calculation procedure consists of analyzing the flow at grid points which are the points of intersection of characteristic lines.

- There can be two types of grid points:
  1. Internal Grid Points
  2. Wall Grid Points



- Lets first analyze the internal grid points as shown. We know the location and flow properties at points 1 and 2.

- Point 3 is simply on the centerline. The C_- characteristic and C_+characteristic are coming from point 2 and 2’ respectively and are symmetric.



$$\theta_2 + \nu_2 = \text{const} = (K_-)_2 = (K_-)_3 \quad \text{along } C_-$$

$$\theta_{2'} - \nu_{2'} = \text{const} = (K_+)_{2'} = (K_+)_3 \quad \text{along } C_+$$

- The constants along the given C_- and C_+ characteristics are the same and opposite $(K_-)_3 = -(K_+)_3$. The flow angle θ_3 is zero since the point is on the centerline. Hence ν_3 can be calculated as:

$$\theta_3 + \nu_3 = \text{const} = (K_-)_3 \quad \text{along } C_- \longrightarrow \nu_3 = (K_-)_3 = -(K_+)_3$$

Internal Points (cont.)

- Using θ_3 and ν_3, we can calculate the remaining flow properties:

Obtain M_3 using ν_3

↓

Calculate p_3 and T_3 from M_3 and previously known p_0, T_0

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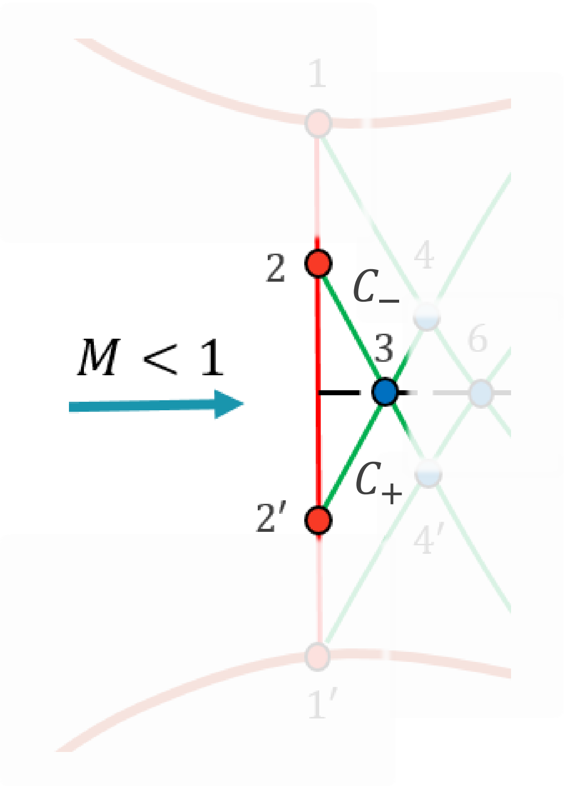
Using T_3, Compute $c_3 = \sqrt{\gamma R T_3}$ and then $V_3 = M_3 c_3$

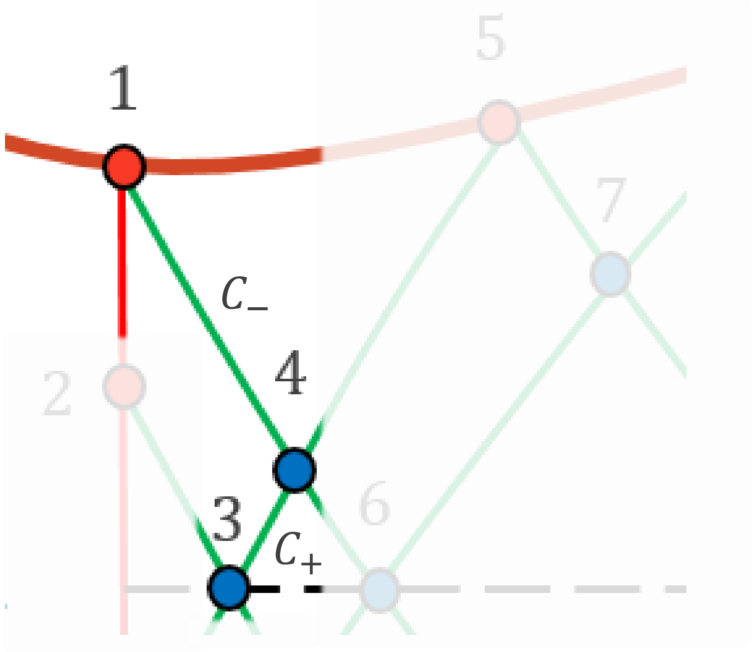


Internal Points (cont.)

- Point 4 is located at the intersection of the C_- characteristic and C_+characteristic through points 1 and 3 respectively. Therefore, we can say:

$$\theta_1 + \nu_1 = \text{const} = (K_-)_4 = (K_-)_1 \quad \text{along } C_-$$

$$\theta_3 - \nu_3 = \text{const} = (K_+)_4 = (K_+)_3 \quad \text{along } C_+$$



- K_− and K_+ are constant along the given C_- and C_+characteristics, and at point 4 we can write:

$$\theta_4 + \nu_4 = (K_-)_4 \qquad \text{along } C_-$$

$$\theta_4 - \nu_4 = (K_+)_4 \qquad \text{along } C_+$$

- Solving these two algebraic equations we obtain the following solution:

$$\theta_4 = \frac{1}{2}\left[(K_-)_1 + (K_+)_3)\right]$$

$$\nu_4 = \frac{1}{2}\left[(K_-)_1 - (K_+)_3)\right]$$

- Using θ_4 and ν_4, we can calculate the remaining flow properties as done for point 3.

Wall Points

- Let's consider an internal point (4) which is close to the wall. The C_+ characteristic through point 4 intersects the wall at point 5.

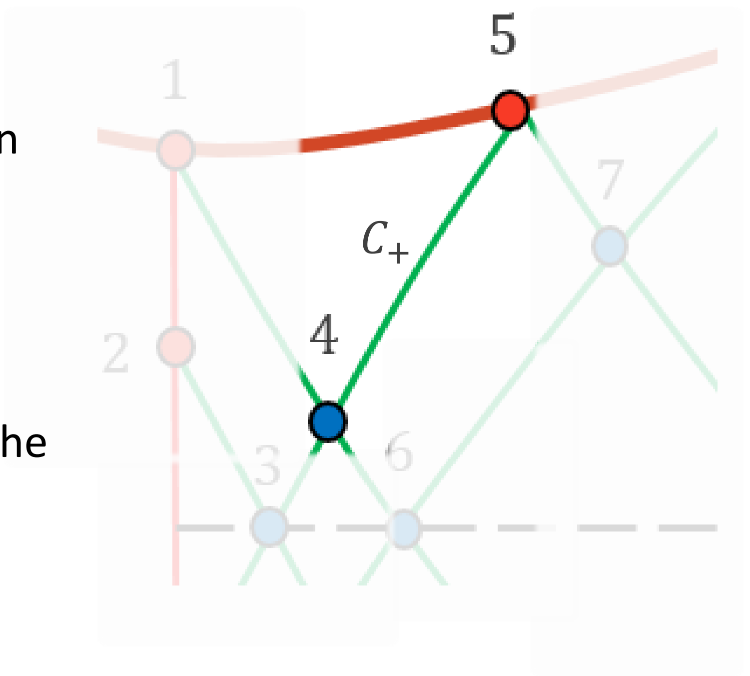
- Assume the slope of the wall at point 5 is known, i.e., θ_5. Then we can obtain the flow properties at this wall point using the properties at point 4 as K_+ is constant along the characteristic C_+:

$$\theta_5 + \nu_5 = \text{const} = (K_+)_4 = (K_+)_5 \qquad \text{along } C_+$$

- Since we already know θ_5 and $(K_+)_5$, we can easily compute ν_5 and follow the procedure outlined in the previous slide to obtain the remaining flow variables.

- Note that for both internal and wall points, our analysis started using the known properties at specific grid points.

- Therefore, we begin our analysis from the *initial data* line and compute the flow properties by marching downstream along the grid defined by the intersection of characteristic lines.



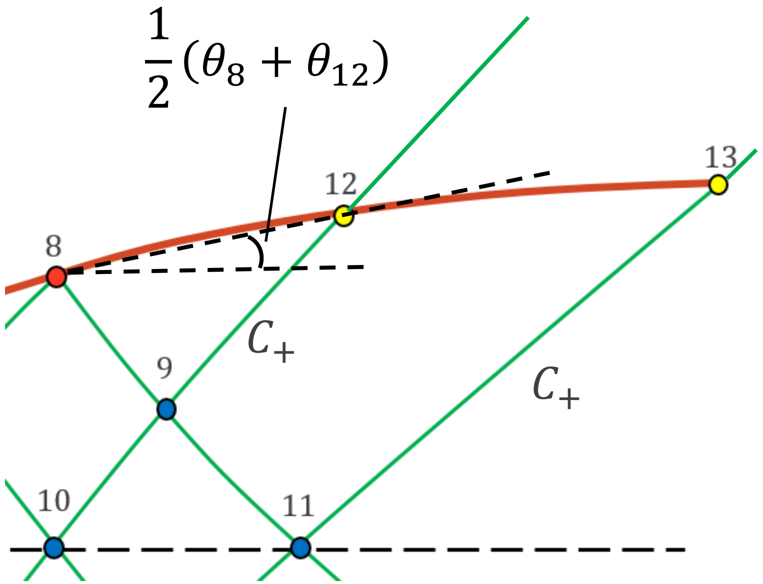
Wall Points (cont.)

- Let's consider the wall points on the straightening section of the nozzle. These points are needed to define the shape of the nozzle profile.

- The slope of the wall at points 12 and 13 is not known in this case.

- However, we know that the straightening section of the nozzle is designed in such way that we have expansion wave cancellations at the wall. Hence, no characteristic line would be generated.

- This means that the flow properties are constant along the characteristic lines C_+ coming from points 9 and 11.



$$\theta_9 = \theta_{12} \text{ and } \nu_9 = \nu_{12}$$

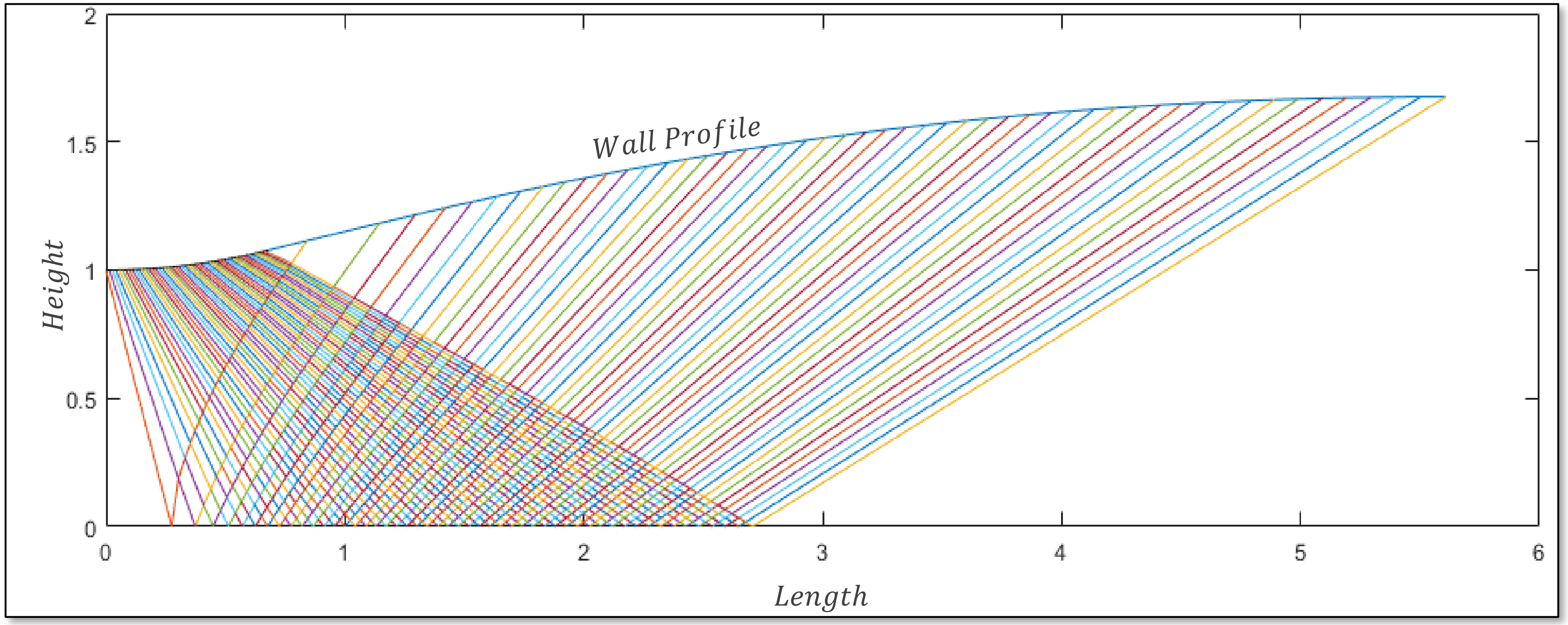
$$\theta_{11} = \theta_{13} \text{ and } \nu_{11} = \nu_{13}$$

- To draw the nozzle profile, start from point 8 and draw a straight line at an angle of $\frac{1}{2}(\theta_8 + \theta_{12})$ that intersects the C_+ line from point 9. The intersection point is point 12.

- Then, using an angle of $\frac{1}{2}(\theta_{12} + \theta_{13})$, repeat the process from point 12 to intersect the C_+ line from point 11 and identify point 13.

Supersonic Nozzle Design

- The procedure outlined in the previous slide can be easily put into a computer program which can be used to compute the optimum profile for a supersonic nozzle.
- The figure below represents the nozzle design for air flow ($\gamma = 1.414$) at a given exit $M = 2.5$ computed using 50 characteristic lines.



Summary

- In this lesson we discussed the method of characteristics.
- We also showed how this method can be used to design the wall profile for a supersonic nozzle.
- The method of characteristics allows us to analyze the flow along the characteristic lines easily as the partial differential equations convert to ordinary differential equations along these lines.
- In the next section, we will cover external compressible flows.

Ansys