Equations of 1D Gas Dynamics

Basics of Compressible Flows – Lesson 3

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Governing Equations of Inviscid Compressible Flow

- Conservation equations of mass, momentum and energy for a control volume in an inviscid gas flow are:

$$\iiint_{\Omega} \frac{\partial \rho}{\partial t}\, d\Omega + \oiint_{A} \rho \vec{V} \cdot \hat{n}\, dA = 0$$

continuity

$$\iiint_{\Omega} \frac{\partial (\rho \vec{V})}{\partial t} d\Omega + \oiint_{A} \vec{V}(\rho \vec{V} \cdot \hat{n}) dA = \iiint_{\Omega} \rho \vec{f}\; d\Omega - \oiint_{A} p\hat{n}\, dA$$

momentum

$$\iiint_{\Omega} \frac{\partial}{\partial t}\left[\rho\left(e + \frac{V^2}{2}\right)\right] d\Omega + \oiint_{A} \rho\left(e + \frac{V^2}{2}\right)(\vec{V} \cdot \hat{n}) dA = \iiint_{\Omega} \rho \dot{q}\, d\Omega - \oiint_{A} p(\vec{V} \cdot \hat{n})\, dA$$

energy

rate of heat added to control volume from surroundings

$$p = \rho R T$$

equation of state

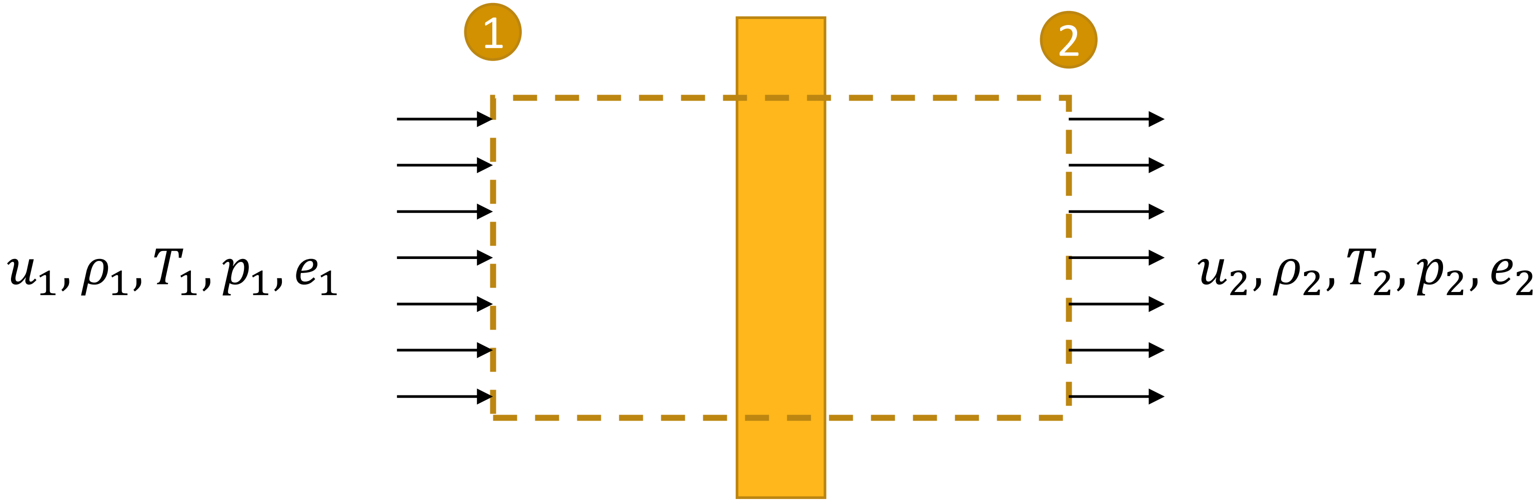
$$e = e(T, v)$$

thermodynamic relation

- We purposely retained the integral form of the governing equations as their differential form is not valid across flow discontinuities (e.g., shocks).

One-Dimensional Analysis

- One-dimensional analysis of incompressible inviscid flows is unexciting, and it yields only a very limited amount of useful information.
- On the other hand, one-dimensional analysis of a compressible flow gives rise to many useful concepts, and we will cover it in this lesson.
- Let us consider a 1D flow from equilibrium state 1 to 2 across the shaded area. We also assume here that the cross-sectional area is constant.



One-Dimensional Steady Equations

- Assuming steady-state flow, the integral conservations laws in 1D reduce to:

$$\rho_1 u_1 = \rho_2 u_2$$

continuity

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2$$

momentum

$$q + \frac{p_1}{\rho_1} - \frac{p_2}{\rho_2} = \left(e_2 + \tfrac{1}{2}u_2^2\right) - \left(e_1 + \tfrac{1}{2}u_1^2\right)$$

energy

- The energy equation can be rewritten in terms of enthalpy, $h = e + p/\rho$:

$$q = \left(h_2 + \tfrac{1}{2}u_2^2\right) - \left(h_1 + \tfrac{1}{2}u_1^2\right)$$

where q is external heat added (per unit mass) to the system through its boundaries.

$$\Downarrow$$

$$h_2 + \tfrac{1}{2}u_2^2 = h_1 + \tfrac{1}{2}u_1^2$$

Adiabatic energy equation when $q = 0$.

- Note that body forces and viscous stresses are neglected in the momentum equation, and the energy equation neglects conduction heat transfer, viscous dissipation and changes in potential energy.
- Interestingly, the energy equation relates conditions at two equilibrium states of the flows, and it will be valid even if there are conduction heat transfer, viscous stresses and other non-equilibrium effects as long as (1) and (2) themselves are equilibrium states.

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Speed of Sound

- For a compressible fluid, the speed at which an acoustic pressure wave propagates will be a significant property when characterizing the flow speed.
- This speed is called the speed of sound, a. For fluids in general, the speed of sound is related to the compressibility β by the Newton-Laplace equation:

$$a^2 = \beta^{-1}\frac{1}{\rho}$$

- The flow through a sound wave is isentropic. Therefore, from isentropic compressibility relation: $\beta = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right)_s$

$$a^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$$

- Thus, for an ideal gas, the speed of sound is given by:

$$a^2 = \left(\frac{\partial p}{\partial \rho}\right)_s = \frac{\gamma p}{\rho} = \gamma RT \Rightarrow a = \sqrt{\gamma RT}$$

The Mach Number

- The ratio of the fluid velocity to the speed of sound is an important dimensionless parameter called the Mach number, M (named for the pioneer in high-speed compressible flow analysis, Ernst Mach):

$$M \equiv \frac{V}{a}$$

- The Mach number allows us to classify compressible flows according to the relationship of the fluid velocity to the speed of sound. This has very important ramifications for the fluid behavior as well as the mathematics of the governing equations.
- The basic classifications are as follows:
  - Incompressible Flow: $M \ll 1$
  - Subsonic Flow: $M < 1$
  - Transonic Flow: $M \sim 1$
  - Supersonic Flow: $M > 1$
  - Hypersonic Flow: $M \gg 1$

Energy Equations Relations

- Recalling the expression of enthalpy for a calorically perfect gas in terms of specific heat, the energy equation can be written as:

$$c_p T_1 + \frac{1}{2}u_1^2 = c_p T_2 + \frac{1}{2}u_2^2$$

$$c_p = \frac{\gamma R}{\gamma - 1} \quad \Downarrow$$

$$\frac{\gamma R T_1}{\gamma - 1} + \frac{1}{2}u_1^2 = \frac{\gamma R T_2}{\gamma - 1} + \frac{1}{2}u_2^2$$

$$a = \sqrt{\gamma R T} \quad \Downarrow$$

$$\frac{a_1^2}{\gamma - 1} + \frac{1}{2}u_1^2 = \frac{a_2^2}{\gamma - 1} + \frac{1}{2}u_2^2$$

$$a = \sqrt{\gamma p/\rho} \quad \Downarrow$$

$$\frac{\gamma}{\gamma - 1}\frac{p_1}{\rho_1} + \frac{1}{2}u_1^2 = \frac{\gamma}{\gamma - 1}\frac{p_2}{\rho_2} + \frac{1}{2}u_2^2$$

Different Forms of the Energy Equation

Total (Stagnation) Conditions

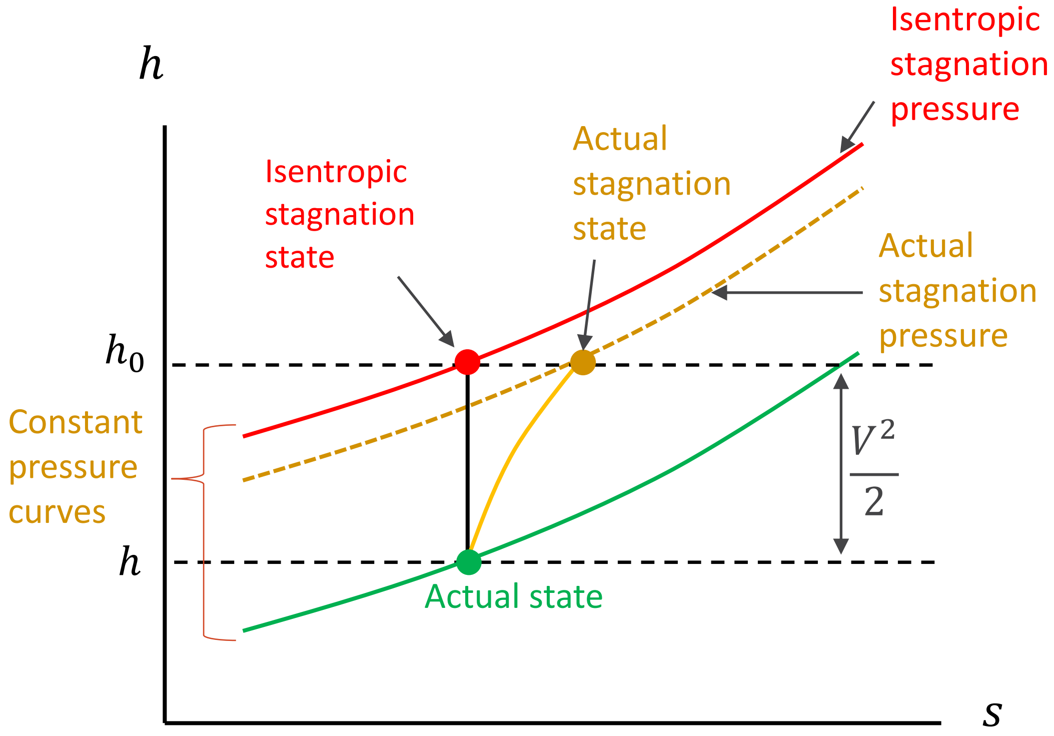
- Consider a compressible gas flow at high speed. If the gas can be slowed down to zero velocity isentropically, it would increase the pressure and temperature along a constant entropy line on an $h - s$ diagram.
- The state achieved by the gas is called the isentropic total state. Note that, due to irreversibility, the actual state is at the same enthalpy but a lower pressure.
- The local total conditions at any point in the gas flow are the conditions that would be reached if the flow were brought to rest isentropically at this point.
- The energy equation gives the following relation for total enthalpy:

$$h_0 = h + V^2/2$$

- From the definition of enthalpy, total temperature for a calorically perfect gas is:

$$T_0 = T + V^2/(2C_p)$$

- Total conditions are commonly denoted by 0 subscript



- Total conditions are also called stagnation conditions as they can be conveniently evaluated at a stagnation point where $V = 0$.
- For a stagnation condition to exist, it is necessary that equilibrium conditions also exist.

Total Relations

- We can now express relations derived from the energy equation in terms of total values.

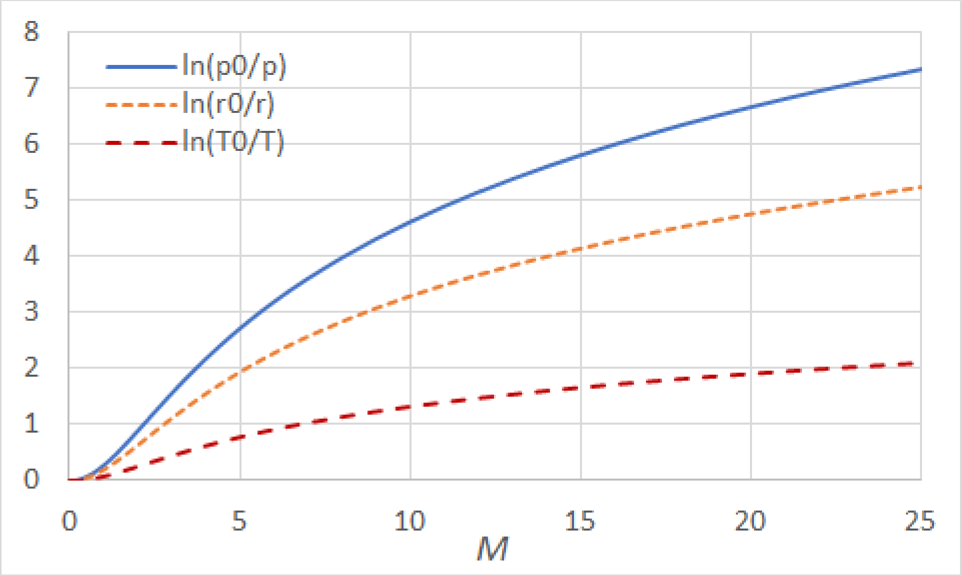
$$c_p T + \frac{1}{2}u^2 = c_p T_0 \quad \Longrightarrow \quad T_0 = T\left(1 + \frac{\gamma - 1}{2} M^2\right)$$

$$c_p = \frac{\gamma R}{\gamma - 1}, a = \sqrt{\gamma R T}$$

- From thermodynamic relations in isentropic processes:

$$p_0 = p\left(1 + \frac{\gamma - 1}{2} M^2\right)^{\gamma/(\gamma-1)}$$

$$\rho_0 = \rho\left(1 + \frac{\gamma - 1}{2} M^2\right)^{1/(\gamma-1)}$$



Sonic Conditions

- In addition to stagnation conditions, for the purpose of analysis it is convenient to define sonic conditions.
- The local sonic condition at any point in the gas flow is the condition that would be reached if the flow were sped up (or slowed down) isentropically until its Mach number is 1.0.
  - We will see later that the sonic condition naturally exists at the throat of a convergent-divergent nozzle.
- Properties at the sonic location are defined with an asterisk superscript:

$$a^* = \sqrt{\gamma R T^*}$$

- In some problems, velocity is scaled by a^* to introduce the characteristic Mach number:

$$M^* = V/a^*$$

- Note this is not the Mach number at the sonic location where it is naturally 1.0, but the Mach number at an arbitrary location scaled by the sonic speed of sound.

Sonic Relations

- Some useful relations between sonic and total conditions:

$$\frac{a^2}{\gamma-1}+\frac{1}{2}u^2=\frac{a_0^2}{\gamma-1} \quad \Longrightarrow \quad \frac{\gamma+1}{2(\gamma-1)}a^{*2}=\frac{a_0^2}{\gamma-1} \quad \underset{a=\sqrt{\gamma RT}}{\Longrightarrow} \quad \left(\frac{a^*}{a_0}\right)^2=\frac{T^*}{T_0}=\frac{2}{\gamma+1}$$

- From isentropic relations for total pressure and density:

$$\frac{p^*}{p_0}=\left(\frac{2}{\gamma+1}\right)^{\gamma/(\gamma-1)} \qquad \frac{\rho^*}{\rho_0}=\left(\frac{2}{\gamma+1}\right)^{1/(\gamma-1)}$$

For air: $T^*/T_0=0.833$, $p^*/p_0=0.528$, $\rho^*/\rho=0.634$

- Finally:

$$\frac{a^2}{\gamma-1}+\frac{1}{2}u^2=\frac{\gamma+1}{2(\gamma-1)}a^{*2} \quad \underset{1/u^2}{\Longrightarrow} \quad M^2=\frac{2}{\left((\gamma+1)/M^{*2}\right)-(\gamma-1)} \quad \Leftrightarrow \quad M^{*2}=\frac{(\gamma+1)M^2}{(\gamma-1)M^2+2}$$

$$\Downarrow$$

$$M^*=1 \Leftrightarrow M=1, \qquad M^*<1 \Leftrightarrow M<1, \qquad M^*>1 \Leftrightarrow M>1$$

$$M^*\rightarrow\sqrt{(\gamma+1)/(\gamma-1)} \Leftrightarrow M\rightarrow\infty$$

- Having M^* finite when M tends to infinity is useful in the analysis of shock discontinuities.

Summary

- In this lesson, we discussed one-dimensional approximations of the governing equations of a gas flow.
- Despite their apparent simplicity, 1D equations can be obtained directly from integral conservation laws, and thus carry all the authority of the original equations.
- The energy equation, when applied to an isentropic process, simply connects its two states without necessarily taking the dimension of a problem into consideration. The energy equation also yielded several useful isentropic relations.

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