Conservation of Momentum Equations

Governing Equations of Fluid Dynamics – Lesson 4

Ansys

Introduction

- We now consider the conservation of momentum equation.
- This is simply a restatement of Newton's Second Law of Motion, which we saw applied to a Lagrangian fluid parcel in Lesson 2.
- Using our technique of applying Reynold's Transport and Divergence theorems, we will show how to obtain the conservation of momentum equation.
- In addition, we will also describe assumptions required to derive viscous stress components of the momentum equation.

Derivation of the Conservation of Momentum Equation

- We again apply Reynold's Transport and Divergence Theorems to the conservation of momentum for the Lagrangian parcel (Newton's 2nd Law):

$$\frac{d\vec{H}}{dt} = \frac{d}{dt}\iiint_{\Omega} \rho\vec{V}\,d\Omega = \iiint_{\Omega} \frac{\partial(\rho\vec{V})}{\partial t}\,d\Omega + \oiint_{A} \vec{V}(\rho\vec{V}\cdot\hat{n})\,dA = \vec{F}_s + \vec{F}_b$$

$$\iiint_{\Omega} \frac{\partial(\rho\vec{V})}{\partial t}\,d\Omega + \oiint_{A} \vec{V}(\rho\vec{V}\cdot\hat{n})\,dA = \vec{F}_s + \vec{F}_b$$

$$\iiint_{\Omega} \left[\frac{\partial(\rho\vec{V})}{\partial t} + \vec{V}\cdot\nabla(\rho\vec{V})\right] d\Omega = \vec{F}_s + \vec{F}_b$$

- To proceed further, we need to define the surface and body forces acting on the fluid.

Surface Forces

- Surface forces acting on the fluid particle are due to pressure and viscous stress.
- The net pressure and viscous stress forces acting on the fluid can be expressed as

$$\vec{F}_s = \vec{F}_p + \vec{F}_v$$

$$\vec{F}_p = -\oiint_A p\hat{n}\, dA$$

pressure force

$$\vec{F}_v = \oiint_A \bar{\bar{\tau}} \cdot \hat{n}\, dA$$

viscous stress force

where $\bar{\bar{\tau}}$ is called the viscous stress tensor.

- Note that pressure acts normal to the surface and the viscous stress, in general, has components that act both normal and tangent to the surface.
- We will discuss the viscous stress tensor later in this lesson.

Volume Integral Forms of Surface Force Terms

- We can transform the pressure force and viscous force terms from a surface integral into a volume integral using the Divergence Theorem :

$$\oiint_A p\hat{n}\, dA = \iiint_\Omega \nabla p\, d\Omega$$

pressure force

$$\oiint_A \bar{\bar{\tau}} \cdot \hat{n}\, dA = \iiint_\Omega \nabla \cdot \bar{\bar{\tau}}\, d\Omega$$

viscous stress

Body Forces

- Body forces act over the entire volume and therefore we can formulate this in terms of a generic body force per unit volume ($\overrightarrow{F_b'}$) integrated over the volume:

$$\vec{F}_b = \iiint_{\Omega} \overrightarrow{F_b'} \; d\Omega$$

- The most common body force is the force due to gravitational acceleration ($\vec{g}$):

$$\vec{F}_b = \iiint_{\Omega} \rho \vec{g} \; d\Omega$$

Derivation of the Conservation of Momentum Equation

- We can now introduce the definitions of the surface and body forces into the momentum conservation equation:

$$\iiint_{\Omega} \left[\frac{\partial (\rho \vec{V})}{\partial t} + \vec{V} \cdot \nabla (\rho \vec{V}) \right] d\Omega = \iiint_{\Omega} \left[-\nabla p + \nabla \cdot \bar{\bar{\tau}} + \vec{F_b'} \right] d\Omega$$

- The control volume is arbitrary; hence the integrand can be set equal to zero. The equation is then reduced to its differential form:

$$\frac{\partial (\rho \vec{V})}{\partial t} + \vec{V} \cdot \nabla (\rho \vec{V}) = -\nabla p + \nabla \cdot \bar{\bar{\tau}} + \vec{F_b'}$$

- Unlike conservation of mass, this is a vector equation, with three Cartesian components associated with the x, y, and z directions.

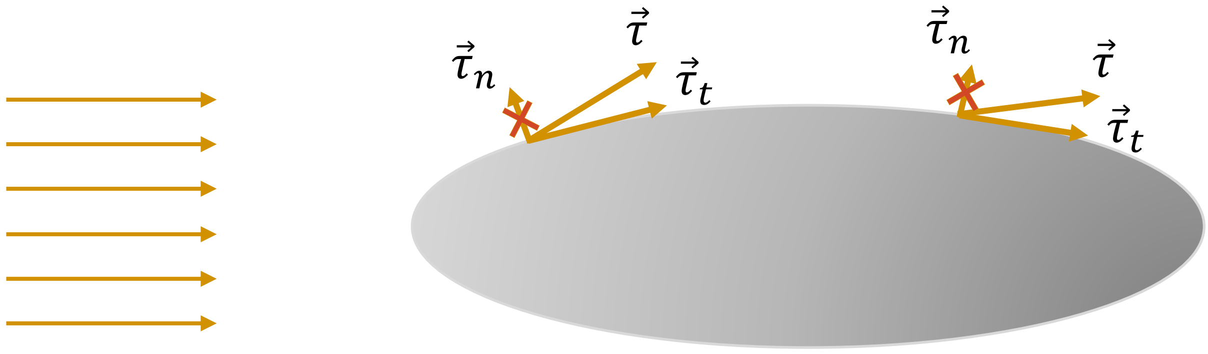
Ansys

Momentum Equation – Viscous Stress Tensor

- In three dimensions, the viscous stress tensor $\bar{\bar{\tau}}$ is a 3 x 3 matrix with nine basic components which are still unknown at this point. For a well-posed mathematical problem, the number of unknowns must be equal to the number of equations. The nine extra unknows would lead to an under-defined problem, and to make the problem well-defined it is required to either:
  - Derive nine additional governing equations for the nine unknow components of the stress tensor, or
  - Express stress tensor components in terms of flow variables, specifically velocity components, to make the problem well-defined without the addition of extra equations
- The second approach of expressing the stress tensor through velocity field variables and their derivatives was first proposed in the first half of the 19^{th} century by Navier, and then Stokes completed the derivations at the end of the 19^{th} century.
- These classical works resulted in the fundamental set of governing equations of fluid dynamics called Navier-Stokes equations.
- Here we are not going into the details of how viscous stresses are derived, but will state main assumptions made by Navier and Stokes, and will describe the final expression for viscous stresses.

Momentum Equation – Viscous Stress Tensor

- Fundamental assumptions made by Navier and Stokes when deriving the viscous stress tensor components are:
  - Pure translation or rigid body rotation of a fluid element does not give rise to any viscous stresses.
  - The fluid flow is isotropic, meaning its properties are the same when measured in different directions.
  - The viscous stress is linearly proportional to the strain rate; they are related by a viscosity tensor not depending on the stress rate and flow velocity (Newtonian fluid assumption).
  - The viscous stress force acts tangentially to the surface of the fluid element, and the normal component of the viscous stress force is zero.



Momentum Equation – Viscous Stress Tensor

- Under the assumptions stated on the previous slide, the final expression of the viscous stress tensor in the Cartesian coordinate system becomes:

$$\bar{\bar{\tau}}_{ij} = 2\mu \left[\varepsilon_{ij} - \frac{1}{3} \nabla \cdot \vec{V}\, \delta_{ij} \right]$$

where

$\varepsilon_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]$ - strain rate tensor describing the rate of change of the fluid element

δ_{ij} - Kronecker delta

μ - dynamic viscosity, which can depend on the local state of fluid

Conservation of Momentum Equations – Final Form

- Finally, by substituting expressions of viscous stresses and taking advantage of the continuity equation, the final differential equations for conservation of momentum become:

$$\rho\left[\frac{\partial \vec{V}}{\partial t}+\vec{V}\cdot\nabla\vec{V}\right]=-\nabla p+\mu\nabla^2\,\vec{V}+\frac{1}{3}\mu\,\nabla\left(\nabla\cdot\vec{V}\right)+\overrightarrow{F_b}$$

- This is the so-called "convective," or "non-conservative," form of the momentum equations.

Conservation of Momentum Equations – Expanded 3D Form

- For subsequent modeling, it is useful to write the equations in expanded 3D form:

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left[2\mu\frac{\partial u}{\partial x}-\frac{2}{3}\mu\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\right]+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right]+F_{b,x}$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial p}{\partial y}+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]+\frac{\partial}{\partial y}\left[2\mu\frac{\partial v}{\partial y}-\frac{2}{3}\mu\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right]+F_{b,y}$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right]+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\right]+\frac{\partial}{\partial z}\left[2\mu\frac{\partial w}{\partial z}-\frac{2}{3}\mu\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\right]+F_{b,z}$$

Expanded 2D Cartesian Form

- For 2D mathematical models, the equations reduce to:

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left[2\mu\frac{\partial u}{\partial x}-\frac{2}{3}\mu\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right]+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+F_{b,x}$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]+\frac{\partial}{\partial y}\left[2\mu\frac{\partial v}{\partial y}-\frac{2}{3}\mu\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right]+F_{b,y}$$

Conservation of Momentum Equations – Conservative Form

- Momentum equations can be recast into the so-called "conservative form":

$$\frac{\partial(\rho u)}{\partial t} + \nabla \cdot (\rho \vec{V} u) = -\frac{\partial p}{\partial x} + \nabla \cdot \left[\mu \left(\frac{\partial \vec{V}}{\partial x} + \nabla u\right) - \frac{2}{3}\mu(\nabla \cdot \vec{V})\ \hat{\imath}\right] + F_{b,x}$$

$$\frac{\partial(\rho v)}{\partial t} + \nabla \cdot (\rho \vec{V} v) = -\frac{\partial p}{\partial y} + \nabla \cdot \left[\mu \left(\frac{\partial \vec{V}}{\partial y} + \nabla v\right) - \frac{2}{3}\mu(\nabla \cdot \vec{V})\ \hat{\jmath}\right] + F_{b,y}$$

$$\frac{\partial(\rho w)}{\partial t} + \nabla \cdot (\rho \vec{V} w) = -\frac{\partial p}{\partial z} + \nabla \cdot \left[\mu \left(\frac{\partial \vec{V}}{\partial z} + \nabla w\right) - \frac{2}{3}\mu(\nabla \cdot \vec{V})\ \hat{k}\right] + F_{b,z}$$

- Conservative and non-conservative forms of the equations are mathematically equivalent only for differentiable (smooth) flow fields.
- For non-differentiable flow fields (e. G., flows with shock discontinuities), the non-conservative form is, strictly speaking, invalid across discontinuities and the conservative form must be used instead.

What Do We Have So Far?

- The conservation of momentum equations, along with conservation of mass, comprise a coupled system of four differential equations with five unknowns: ρ, p, u, v, w.
- If the flow is incompressible, then density is constant and we can simplify the unknowns to p, u, v, w. If we further assume constant viscosity, we have a complete set of equations, and simply need to provide initial/boundary conditions in order to develop a solvable mathematical model.
- However, if density and viscosity are NOT constant (and, in fact are, functions of temperature), then we will need at least one more equation to close the system. If temperature is required, then we will need the energy equation, which will be considered in the next lesson. If the fluid is also compressible, then we will also need an equation of state which relates density to pressure and temperature (for example, through the ideal gas law).

Summary

- We have considered the derivation of the conservation of momentum equations in this lesson.
- Although the derivation used some advanced mathematical concepts, the resulting differential equations are straightforward and can be expressed as a coupled system of three differential equations.
- Next, we will take up the important conservation of energy equation.

Ansys