Electromagnetic Plane Waves

Sources

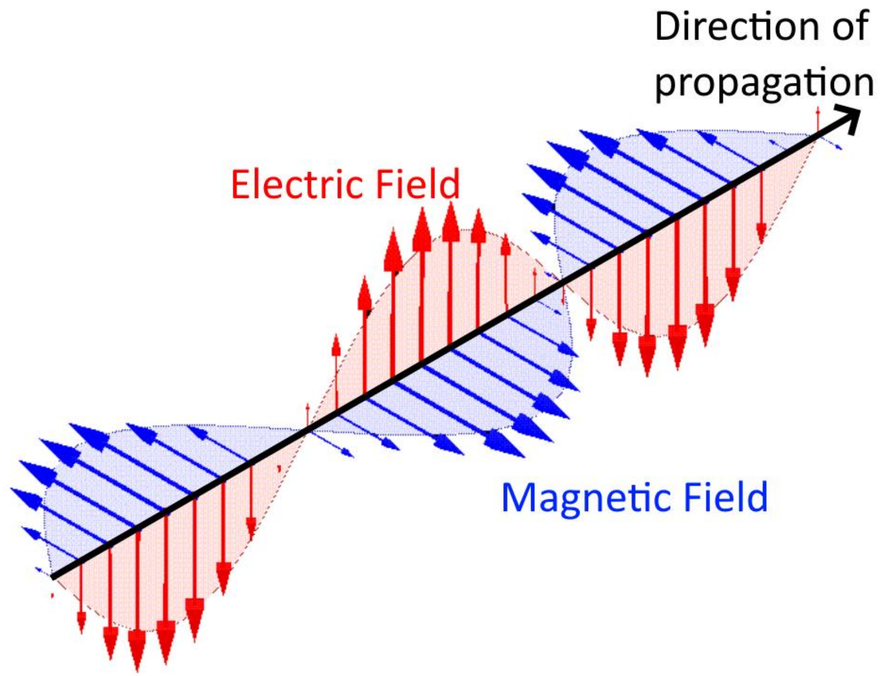
The material presented herein is from the following sources:

"Elements of Electromagnetics," by Matthew N.O Sadiku, 5th ed. (2010)
"Engineering Electromagnetics," by Nathan Ida, 3rd ed. (2015)
"Microwave Engineering," by David Pozar, 4th ed. (2012)

What is a Plane Wave?

A plane wave is a transverse electromagnetic wave that is constant, in both magnitude and direction, over a plane normal to the direction of propagation

A transverse electromagnetic, or TEM, wave, is a wave where the electric field $\overline{E}$ and magnetic field $\overline{H}$ are perpendicular both to one another, and to the direction of propagation.

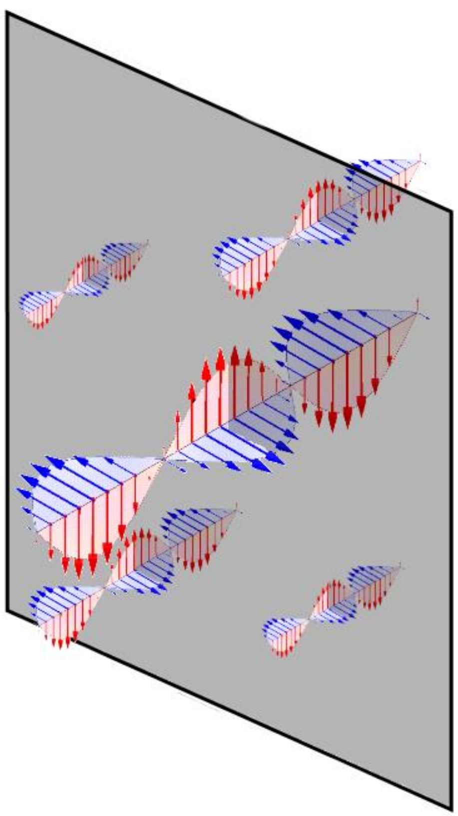


A TEM wave

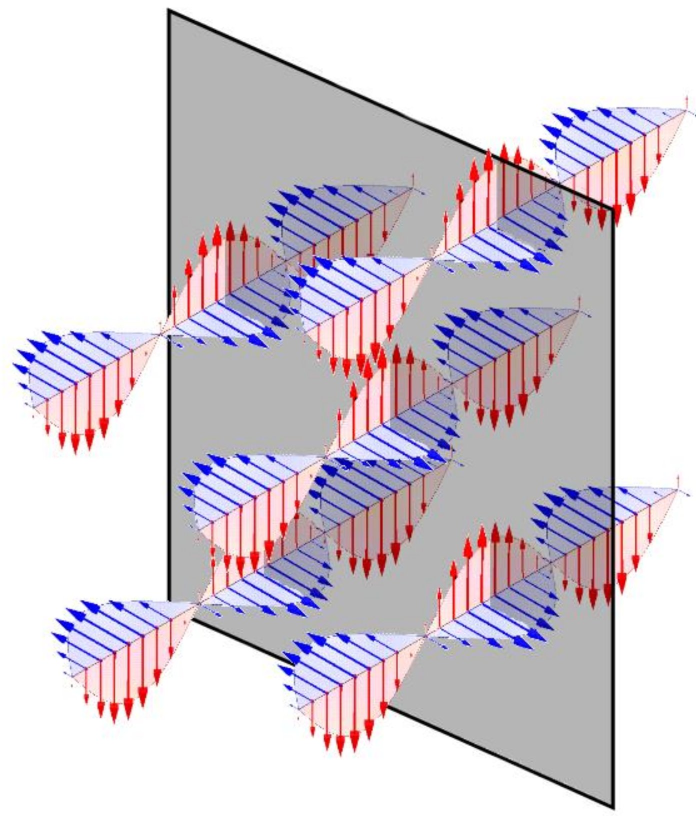
What is a Plane Wave?

A plane wave is a transverse electromagnetic wave that is constant, in both magnitude and direction, over a plane normal to the direction of propagation

If you look at an electromagnetic wave on a plane normal to the direction of propagation…



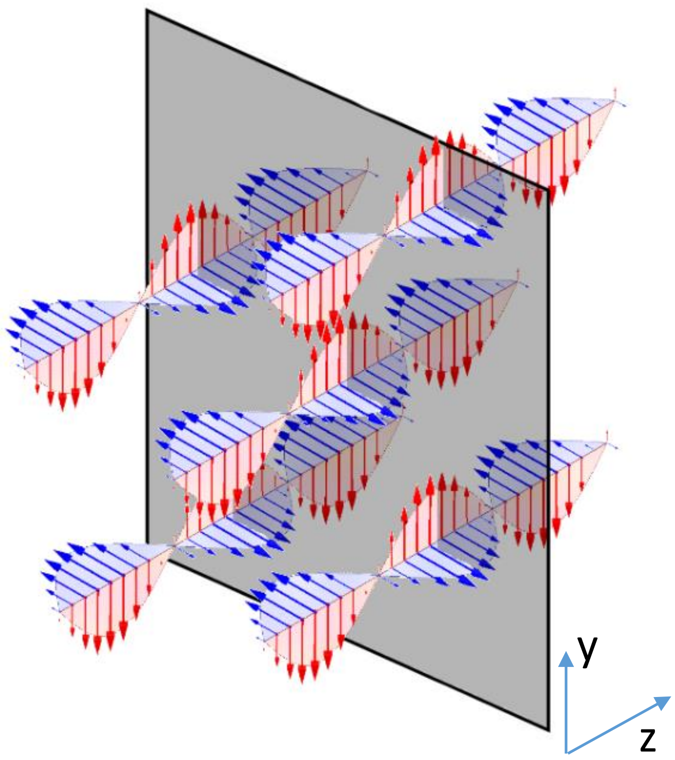
A TEM wave may vary over the plane (in field direction or magnitude)



A plane wave is constant over that plane.

What is a Plane Wave?

A plane wave is a transverse electromagnetic wave that is constant, in both magnitude and direction, over a plane normal to the direction of propagation



Consider: A TEM wave has the following electric field:

$$\overline{E} = E_o e^{-j2z}\widehat{a_y}$$ *(phasor domain)*

$$\overline{E} = E_o \cos(\omega t - 2z)\widehat{a_y}$$ *(time domain)*

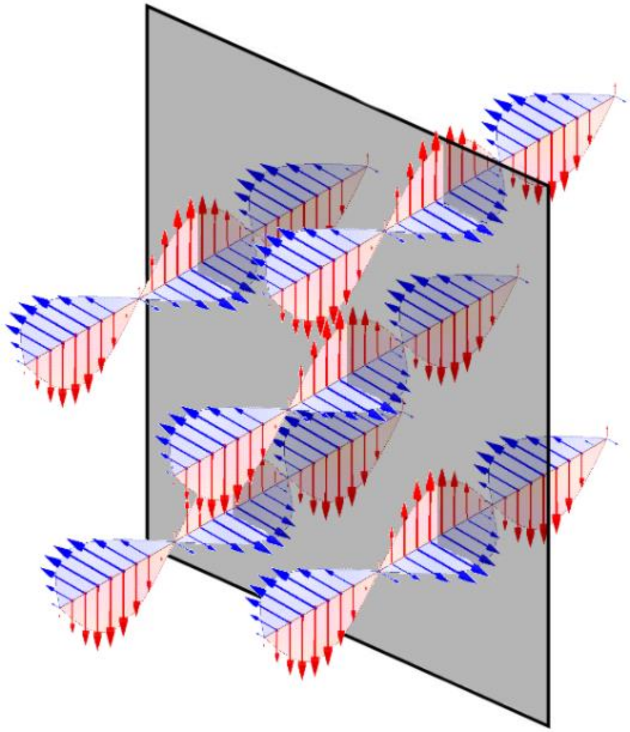
from which we may observe that:

- The wave is propagating in the +z direction
- The electric field is oriented along the y-axis
- Since the wave is TEM, the magnetic field will be oriented along the x-axis
- E_o is the magnitude of the electric field vector.

Note: If E_o does not vary as a function of x or y, this equation represents a plane wave.

What is a Plane Wave?

Plane waves are mathematically convenient constructs we can use to derive general principles about electromagnetic waves.



Notes:

Plane waves are *simplified idealizations* of electromagnetic waves (no real wave is perfectly constant over a plane).

Waves may often be *approximated* as plane waves with great accuracy (as when far from the source).

Electromagnetic wave properties are often derived with respect to plane waves, because of their mathematical simplicity. However, the principles derived from plane wave analysis are often generalizable to all EM waves.

Intrinsic Impedance of Plane Waves

A plane wave has a fixed ratio of electric field magnitude to magnetic field magnitude.

Recall, the source-free wave equations for electric and magnetic fields were given by:

$$\nabla^2 \overline{E} - k^2 \overline{E} = 0$$

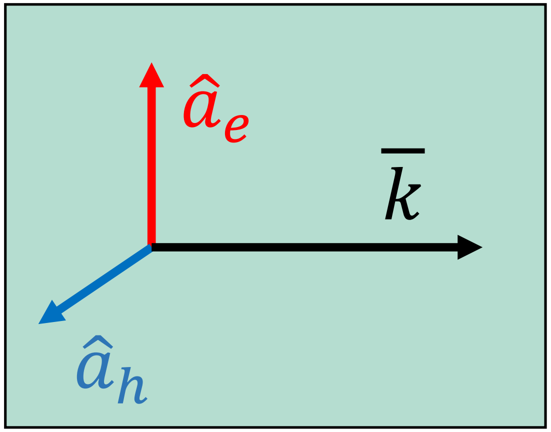
$$\nabla^2 \overline{H} - k^2 \overline{H} = 0$$

which have plane-wave solutions of the form:

$$\overline{E} = \left[E_o{}^{+} e^{-j\overline{k}\cdot\overline{r}} + E_o{}^{-} e^{+j\overline{k}\cdot\overline{r}} \right] \hat{a}_e$$

$$\overline{H} = \left[H_o{}^{+} e^{-j\overline{k}\cdot\overline{r}} + H_o{}^{-} e^{+j\overline{k}\cdot\overline{r}} \right] \hat{a}_h$$

where we know that $\hat{a}_e$, $\hat{a}_h$, and $\overline{k}$ form a right-handed triad:



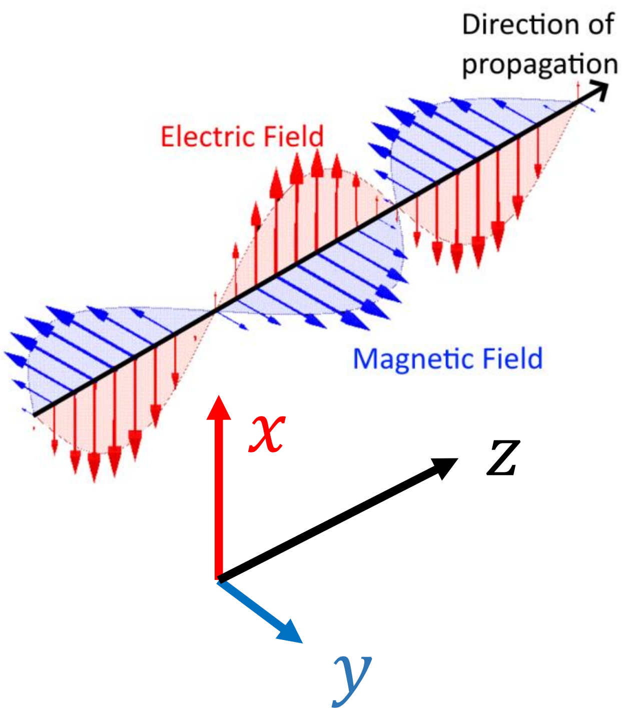
Intrinsic Impedance of Plane Waves

A plane wave has a fixed ratio of electric field magnitude to magnetic field magnitude.

So, for a plane wave propagating in the ± z-direction, with electric field oriented along the x-axis, we have

$$\overline{E} = \left[E_x{}^{+} e^{-jkz} + E_x{}^{-} e^{+jkz}\right]\hat{a}_x$$

$$\overline{H} = \left[H_y{}^{+} e^{-jkz} + H_y{}^{-} e^{+jkz}\right]\hat{a}_y$$



But by Faraday's Law, we also have:

$$\overline{H} = -\frac{1}{j\omega\mu}\nabla \times \overline{E}$$

$$\overline{H} = \frac{k}{\omega\mu}\left[E_x{}^{+} e^{-jkz} - E_x{}^{-} e^{+jkz}\right]\hat{a}_y$$

which we may equate to the expression for $\overline{H}$ above to obtain the following two relations:

$$E_x{}^{+} = \frac{\omega\mu}{k} H_y{}^{+}$$

$$E_x{}^{-} = -\frac{\omega\mu}{k} H_y{}^{-}$$

Intrinsic Impedance of Plane Waves

A plane wave has a fixed ratio of electric field magnitude to magnetic field magnitude.

Now we are ready to define a new term. The "Intrinsic Impedance" of a wave is given by:

$$\eta = \frac{E_x^{\;+}}{H_y^{\;+}} = -\frac{E_x^{\;-}}{H_y^{\;-}} = \frac{\omega\mu}{k} = \sqrt{\frac{\mu}{\varepsilon}}$$

Or, in words,

$$\text{intrinsic impedance} = \frac{\text{Forward E}}{\text{Forward H}} = -\frac{\text{Backward E}}{\text{Backward H}} = \sqrt{\frac{\mu}{\varepsilon}}$$

Note: in free space,

$$\eta_o = \sqrt{\frac{\mu_o}{\varepsilon_o}} = 377\Omega$$

Intrinsic Impedance of Plane Waves

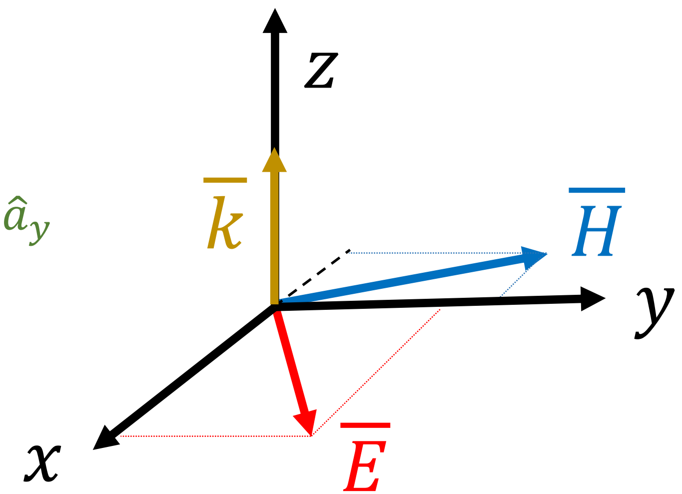
A plane wave has a fixed ratio of electric field magnitude to magnetic field magnitude.

Using the intrinsic impedance, if you're told that you're dealing with a TEM wave, and you know the electric field, you can write the magnetic field directly, and vice versa.

Ex) If the electric field of a TEM wave is given by:

$$\overline{E} = E_x{}^{+} e^{-jkz} \hat{a}_x + E_y{}^{+} e^{-jkz} \hat{a}_y$$

what is its magnetic field?



<u>Solution:</u>

Propagation is in +z, so by the right-hand rule, the x-component of the electric field will contribute a y-component to the magnetic field. The y-component of the electric field will contribute a negative x-component to the magnetic field. The ratios are dictated by the characteristic impedance, so…

$$\overline{H} = \frac{E_y{}^{+}}{\eta} e^{-jkz} \hat{a}_x - \frac{E_x{}^{+}}{\eta} e^{-jkz} \hat{a}_y$$

Material Interaction with Plane Waves

A plane wave propagating in a dielectric will have different properties than a plane wave propagating in free space…

Let’s look back again at the source-free wave equations:

$$\nabla^2 \overline{E} - k^2 \overline{E} = 0$$

$$\nabla^2 \overline{H} - k^2 \overline{H} = 0$$

Here, the wavenumber *k* is given by:

$$k = \omega\sqrt{\mu\epsilon}$$

from which we obtain that the phase velocity of the wave is:

$$v_{ph} = \frac{1}{\sqrt{\mu\epsilon}}$$

In free space, this works out to:

$$v_{ph} = \frac{1}{\sqrt{\mu_o\epsilon_o}} = 3 \times 10^8\ m/s$$

but in a material, we must consider the possibility of $\mu \neq \mu_o$, $\epsilon \neq \epsilon_o$.

Material Interaction with Plane Waves

A plane wave propagating in a dielectric will have different properties than a plane wave propagating in free space…

The response of a material to an electromagnetic field is modeled using the material properties of magnetic permeability (μ) and electric permittivity (ϵ). In past modules, we have introduced the notation:

$$\mu = \mu_r \mu_o, \epsilon = \epsilon_r \epsilon_o$$

where μ_r is the “relative permeability” and ϵ_r is the “relative permittivity” of the material.

Now we will expand this model using the following notation, which allows us to account for dielectric losses in the material:

$$\mu = \mu_r \mu_o$$

$$\epsilon = \epsilon' + j\epsilon''$$

Material Interaction with Plane Waves

A plane wave propagating in a dielectric will have different properties than a plane wave propagating in free space…

$$\epsilon = \epsilon' + j\epsilon''$$

In this equation, the real part is the lossless permittivity, like we had before:

$$\epsilon' = \epsilon_r \epsilon_o$$

The imaginary component accounts for dielectric losses, which are introduced by conductivity (σ) in the material according to the following relation:

$$\epsilon'' = \frac{\sigma}{\omega}$$

We will also define the **loss tangent**, as:

$$\tan \delta = \frac{\epsilon''}{\epsilon'} = \frac{\sigma}{\omega \epsilon_r \epsilon_o}$$

Material Interaction with Plane Waves

A plane wave propagating in a dielectric will have different properties than a plane wave propagating in free space…

$$\epsilon = \epsilon' + j\epsilon''$$

This means that, to account for the possibility of losses, we must use the complex propagation constant γ, defined by:

$$\gamma = jk = j\omega\sqrt{\mu(\epsilon' + j\epsilon'')} = \alpha + j\beta$$

Note that, here, α is the **attenuation constant**, and β is the **lossless propagation constant.**

Let's look at how this plays out in terms of a wave…

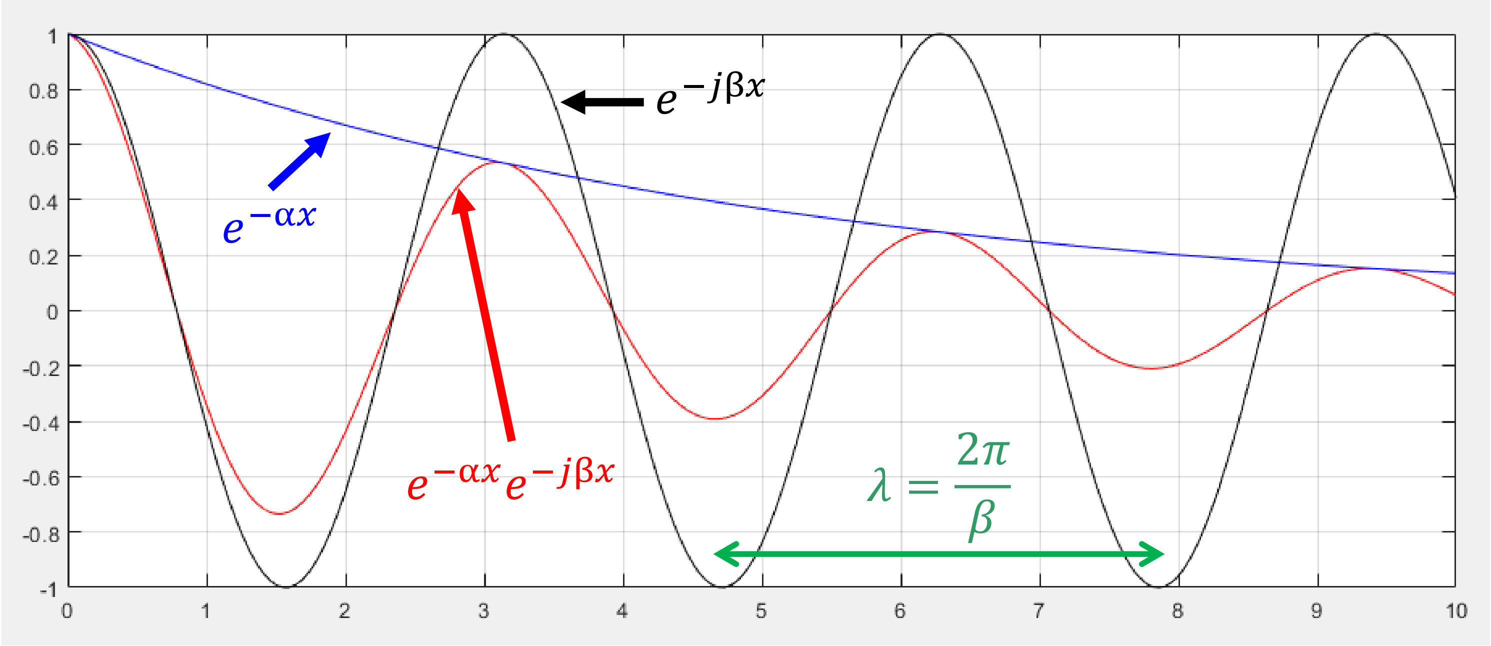
Material Interaction with Plane Waves

A plane wave propagating in a dielectric will have different properties than a plane wave propagating in free space…

A plane wave propagating in +x through a lossy medium, with electric field oriented in z, is described by:

$$\overline{E} = E_o e^{-\gamma x}\,\hat{a}_z = E_o e^{-(\alpha + j\beta)x}\hat{a}_z = E_o\, e^{-\alpha x} e^{-j\beta x}\hat{a}_z$$

This is a sinusoidal function, according to $e^{-j\beta x}$, that decays at a rate of $e^{-\alpha x}$.



Polarization of Plane Waves

Polarization *describes the* ***time-varying*** *orientation of the* ***electric*** *field.*

The polarization of a plane wave is categorized according to the figure traced by the tip of the electric field vector over time, on a fixed plane normal to the direction of propagation.

All plane waves exhibit one of the following three categories of polarization:

1) Linear Polarization
2) Circular Polarization
3) Elliptical Polarization

Polarization of Plane Waves

To determine the polarization of a plane wave, perform the following steps:

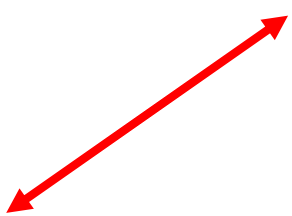
1) Make sure you are looking at the **time-domain** representation of the **electric** field.

2) Choose a convenient observation plane **normal** to the direction of propagation.

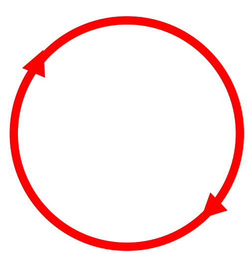
3) Choose a moment in time, and plot the electric field vector on the observation plane at that moment.

4) Determine how increasing time will change the electric field.

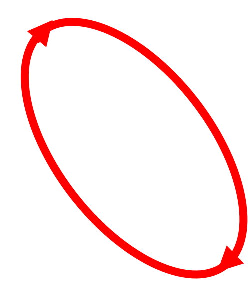
5) If the tip of the electric field traces out a line, the wave is **linearly polarized**. If it will trace a circle, the wave is **circularly polarized**, and if it traces an ellipse, the wave is **elliptically polarized.**



A Linear Trace



A Circular Trace



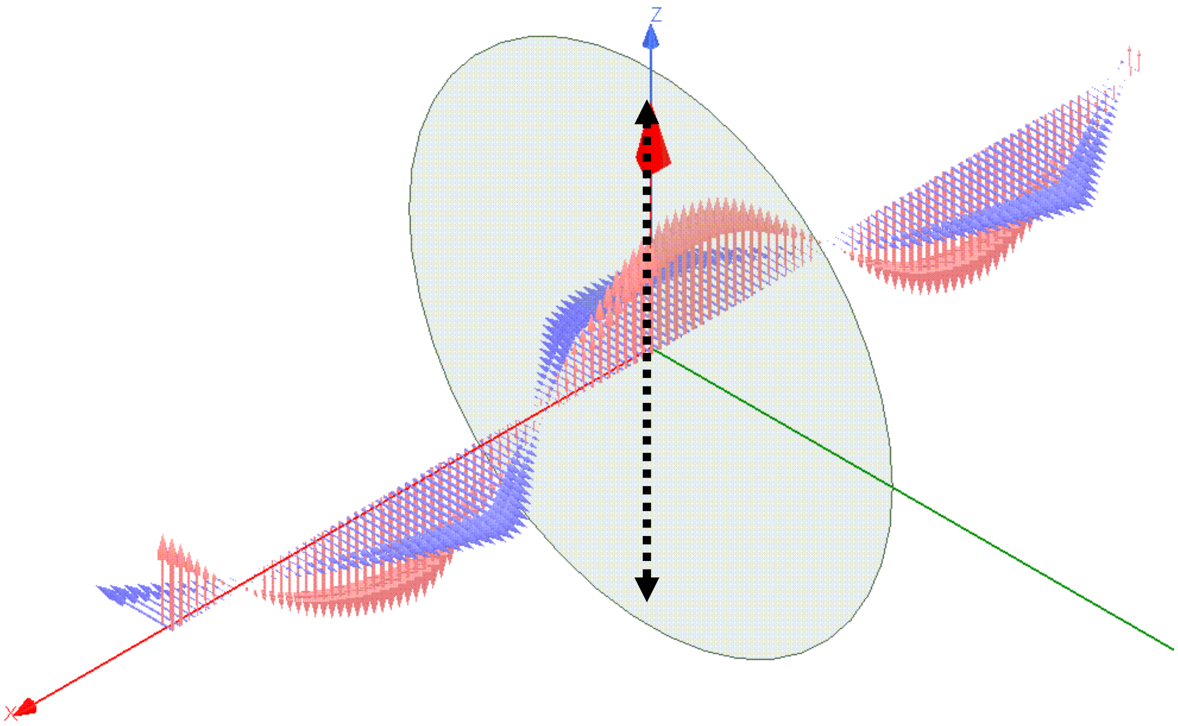
An Elliptical Trace

Linear Polarization

Linear Polarization *occurs when the electric field varies in only a single direction.*

For example, if a plane wave has an electric field given by:

$$\overline{E} = E_o \cos(\omega t - 2x)\widehat{a_z}$$



This electric field, viewed on any plane perpendicular to the direction of propagation, will vary only up and down along the z-axis

Linear Polarization

Linear Polarization *occurs when the electric field varies in only a single direction.*

$$\overline{E} = E_o \cos(\omega t - 2x)\widehat{a_z}$$

Notes: In order for a plane wave to exhibit **linear** polarization, it must have an electric field that **either**

1) Varies in only a single direction

Ex) $\overline{E} = E_z \cos(\omega t - 2x)\widehat{a_z}$

<u>or</u>

2) Consists of two orthogonal components that are out of phase by some multiple of ± 180°

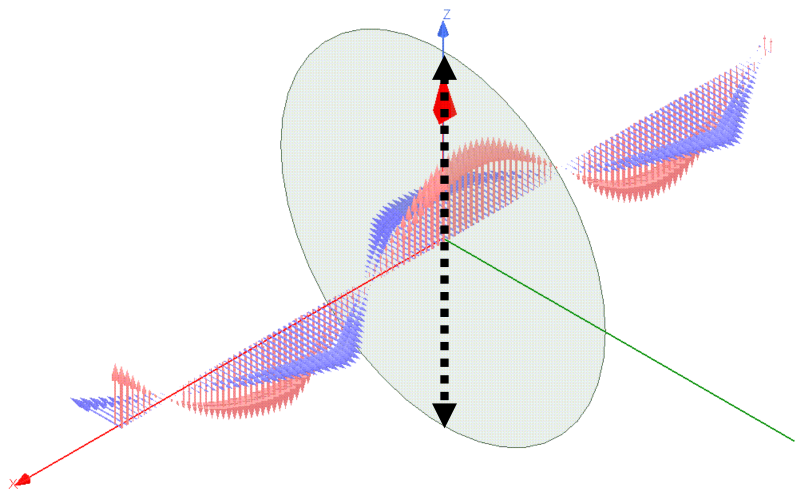
Ex) $\overline{E} = E_y \cos(\omega t - 2x)\widehat{a_y} + E_z \cos(\omega t - 2x + n\pi)\widehat{a_z}$

Linear Polarization

Linear Polarization *occurs when the electric field varies in only a single direction.*

Notes: A linearly polarized wave may be further categorized according to the orientation of the traced line. For example, this wave is linearly polarized **along the z-axis.**

$$\overline{E} = E_o \cos(\omega t - 2x)\widehat{a_z}$$

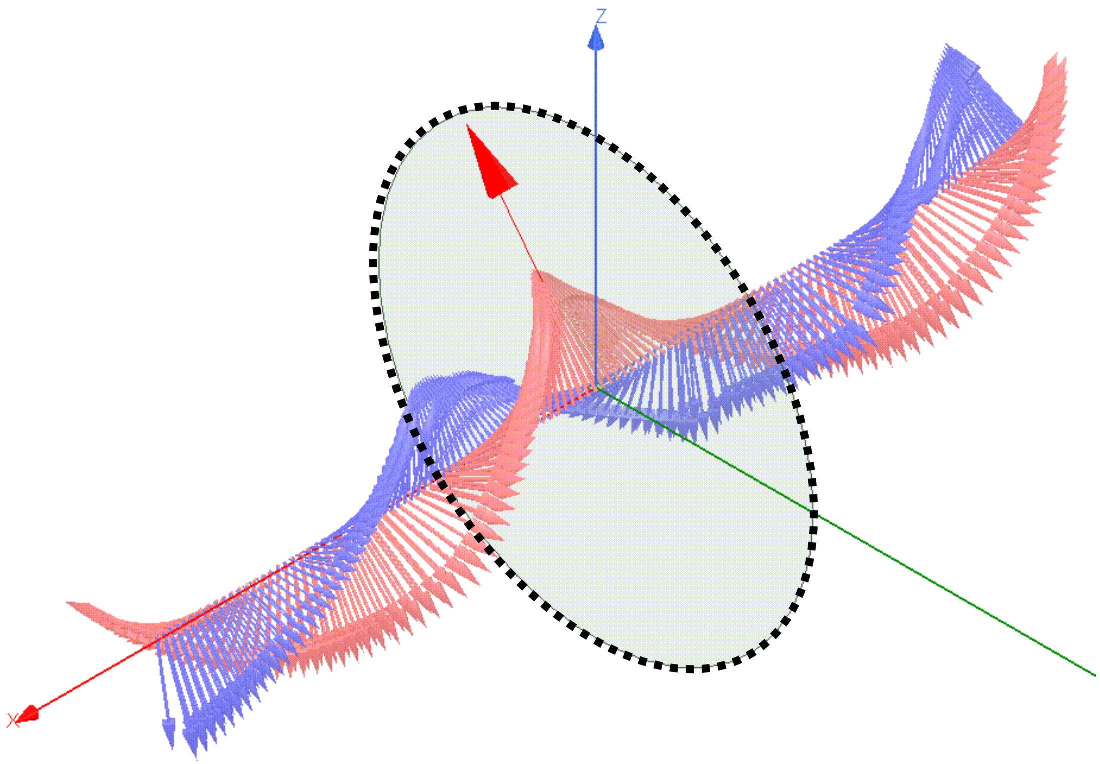


Circular Polarization

Circular Polarization *occurs when the tip of the electric field vector traverses a circle on a plane normal to the direction of propagation.*

For example, if a plane wave has an electric field given by:

$$\overline{E} = E_o \cos(\omega t - 2x)\widehat{a_y} + E_o \cos(\omega t - 2x + \frac{\pi}{2})\widehat{a_z}$$



The tip of the electric field vector, viewed on any plane perpendicular to the direction of propagation, will traverse a circular path as the wave propagates.

Circular Polarization

Circular Polarization *occurs when the tip of the electric field vector traverses a circle on a plane normal to the direction of propagation.*

$$\overline{E} = E_o\cos(\omega t - 2x)\widehat{a_y} + E_o\cos(\omega t - 2x + \frac{\pi}{2})\widehat{a_z}$$

Notes: In order for a plane wave to exhibit **circular** polarization,

1) It must consist of **two** orthogonal components

Ex) $\overline{E} = E_o\cos(\omega t - 2x)\widehat{a_y} + E_o\cos(\omega t - 2x + \frac{\pi}{2})\widehat{a_z}$

2) The two orthogonal components must be ±**90º** out of phase

Ex) $\overline{E} = E_o\cos(\omega t - 2x)\widehat{a_y} + E_o\cos(\omega t - 2x + \frac{\pi}{2})\widehat{a_z}$

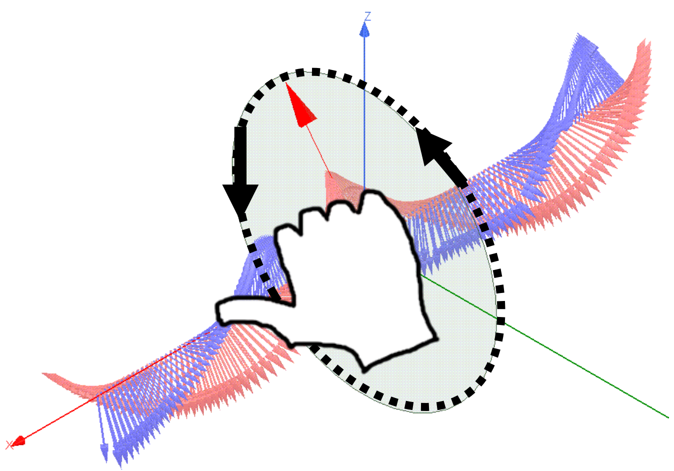
3) The two orthogonal components must be **equal in magnitude**

Ex) $\overline{E} = E_o\cos(\omega t - 2x)\widehat{a_y} + E_o\cos(\omega t - 2x + \frac{\pi}{2})\widehat{a_z}$

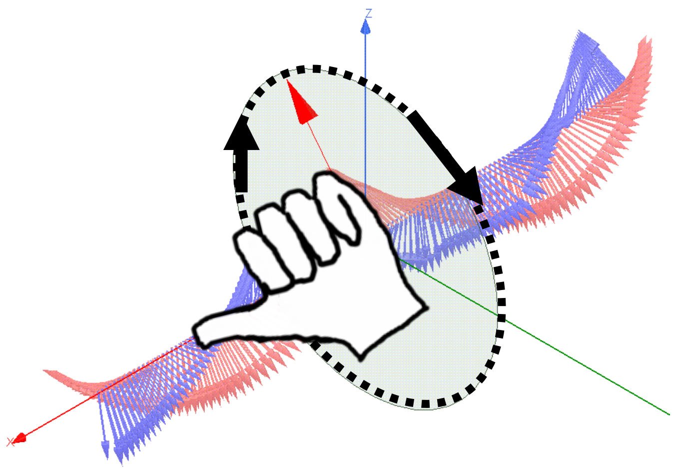
Circular Polarization

Circular Polarization *occurs when the tip of the electric field vector traverses a circle on a plane normal to the direction of propagation.*

Notes: A circularly polarized wave may be further categorized according to the handedness of its rotation. To determine handedness, point your right-hand thumb in the direction of propagation; if your fingers curl in the direction of field rotation, the wave is right-handed. If your fingers curl against the direction of field rotation, the wave is left-handed. For example…



This wave, propagating in +x, is right-handed.



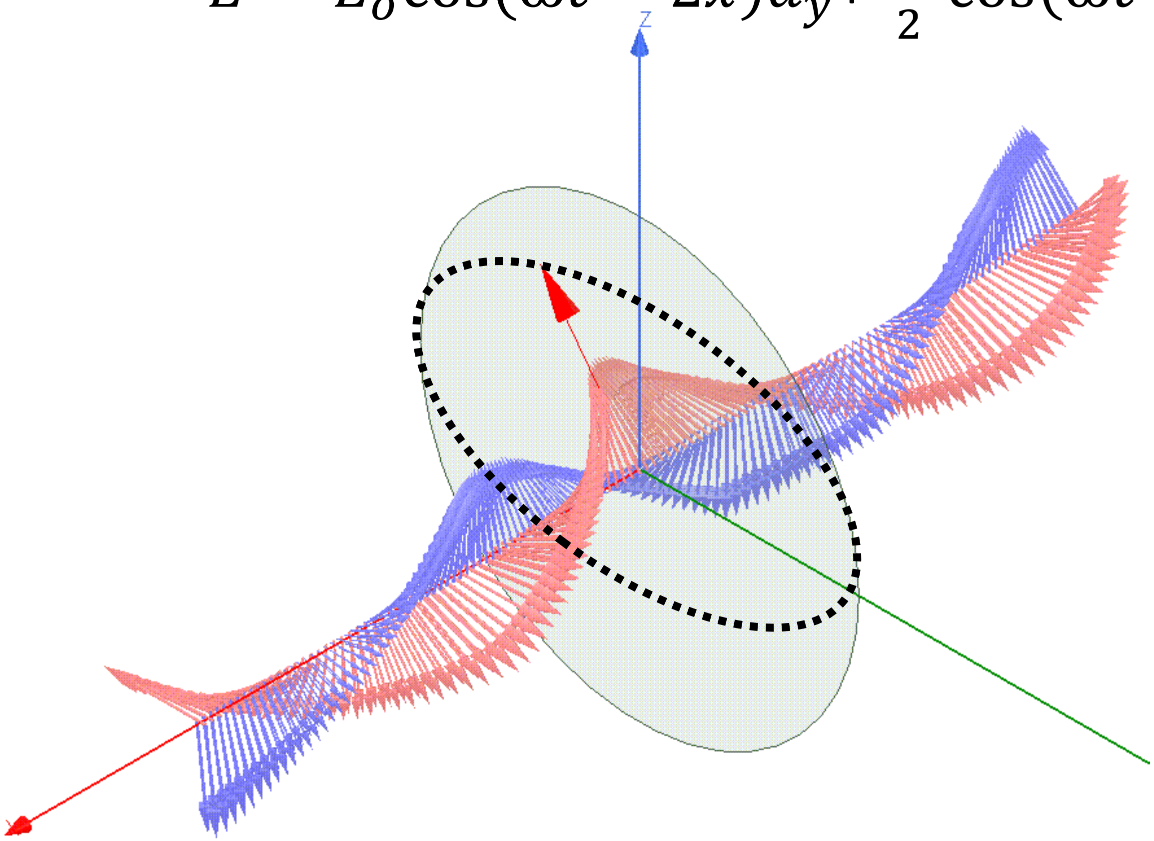
This wave, propagating in +x, is left-handed.

Elliptical Polarization

Elliptical Polarization *occurs when the tip of the electric field vector traverses an ellipse on a plane normal to the direction of propagation.*

For example, if a plane wave has an electric field given by:

$$\overline{E} = E_o \cos(\omega t - 2x)\widehat{a_y} + \frac{E_o}{2}\cos(\omega t - 2x + \frac{\pi}{2})\widehat{a_z}$$



The tip of the electric field vector, viewed on any plane perpendicular to the direction of propagation, will traverse an elliptical path as the wave propagates.

Elliptical Polarization

Elliptical Polarization *occurs when the tip of the electric field vector traverses an ellipse on a plane normal to the direction of propagation.*

$$\overline{E} = E_o\cos(\omega t - 2x)\widehat{a_y} + \frac{E_o}{2}\cos(\omega t - 2x + \frac{\pi}{2})\widehat{a_z}$$

Notes: In order for a plane wave to exhibit **elliptical** polarization,

1) It must consist of **two** orthogonal components

Ex) $\overline{E} = E_o\cos(\omega t - 2x)\widehat{a_y} + \frac{E_o}{2}\cos(\omega t - 2x + \frac{\pi}{2})\widehat{a_z}$

2) The two orthogonal components must be **out of phase**

Ex) $\overline{E} = E_o\cos(\omega t - 2x)\widehat{a_y} + \frac{E_o}{2}\cos(\omega t - 2x + \frac{\pi}{2})\widehat{a_z}$

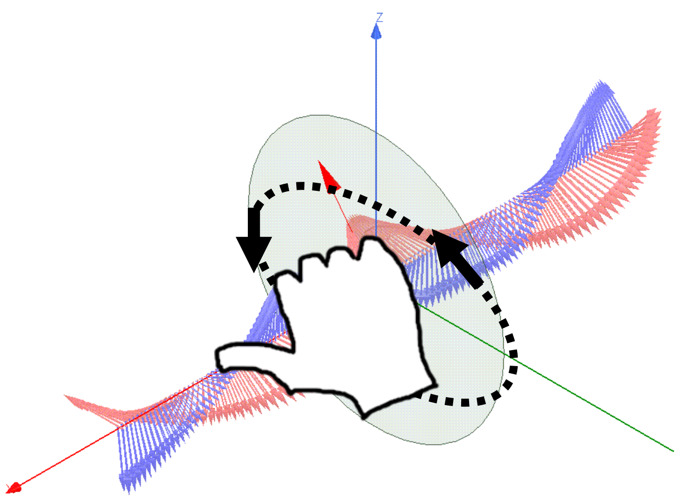
3) The wave must **not** fit the criteria for linear or circular polarization

Ex) $\overline{E} = E_o\cos(\omega t - 2x)\widehat{a_y} + \frac{E_o}{2}\cos(\omega t - 2x + \frac{\pi}{2})\widehat{a_z}$

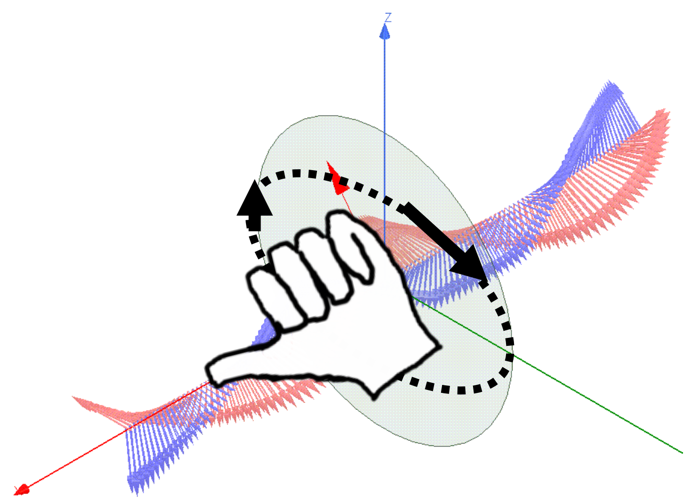
Elliptical Polarization

Elliptical Polarization *occurs when the tip of the electric field vector traverses an ellipse on a plane normal to the direction of propagation.*

Notes: Like circularly polarized waves, elliptically polarized waves may be categorized according to the handedness of their rotation.



This wave, propagating in +x, is right-handed.



This wave, propagating in +x, is left-handed.

Plane Waves at a Boundary

When a plane wave is incident upon a planar boundary between two media, the resulting reflections and transmissions are governed by field boundary conditions.

Recall, at a boundary between two dielectrics:

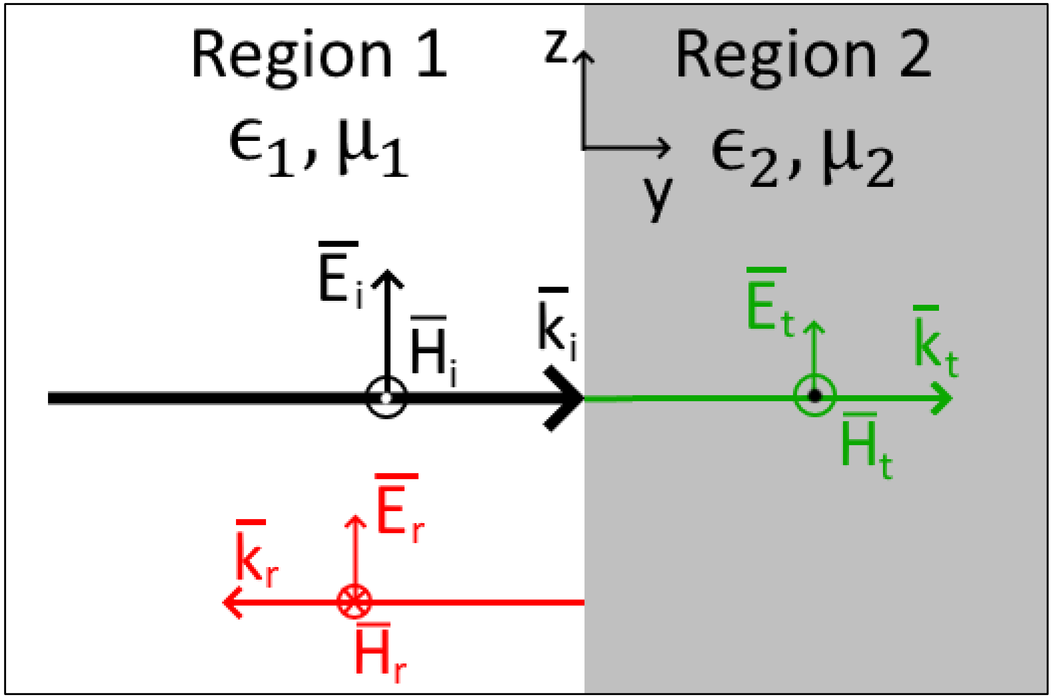
$$E_{1t} = E_{2t}$$
$$D_{1n} - D_{2n} = \rho_s$$
$$B_{1n} = B_{2n}$$
$$\overline{H}_{1t} - \overline{H}_{2t} = \overline{J}_s$$

These rules will govern the behavior of EM waves at a planar boundary.
We will consider this scenario in three cases – all cases of electromagnetic plane waves impinging on a boundary between two dielectrics may be considered as a linear sum of these three.

Plane Waves at a Boundary

Case 1: Normal Incidence

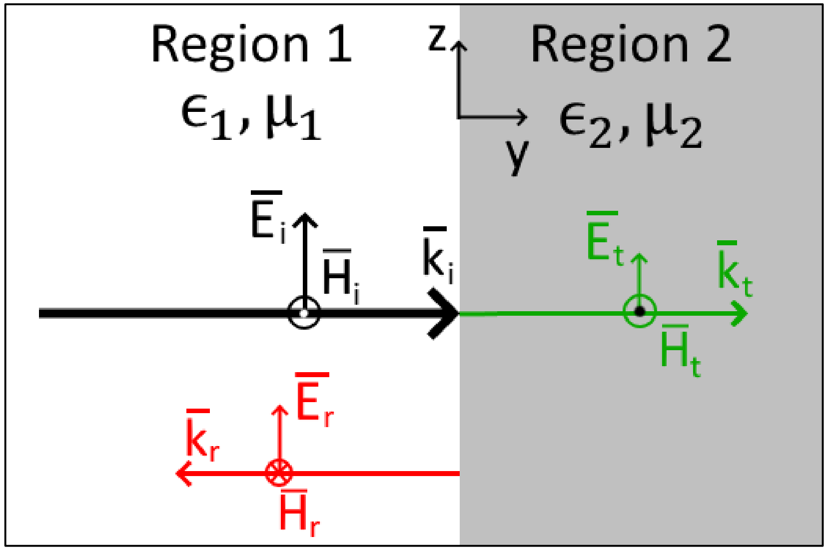
Suppose a plane wave $\overline{E}_i = E_{oi}e^{-\gamma_1 y}\hat{a}_z$ is incident upon a planar boundary, at an angle normal to the surface of the boundary:



In this case, the incident electric and magnetic fields $\overline{E}_i$ and $\overline{H}_i$ are both perfectly tangential to the boundary. Therefore, the transmitted and reflected electric and magnetic fields will also be entirely tangential.

Plane Waves at a Boundary

Case 1: Normal Incidence



From the boundary condition

$$E_{1t} = E_{2t}$$

we can write:

$$\bar{E}_i + \bar{E}_r = \bar{E}_t$$

or:

$$E_{oi} e^{-j\gamma_1 y} \hat{a}_z + E_{or} e^{+j\gamma_1 y} \hat{a}_z = E_{ot} e^{-j\gamma_2 y} \hat{a}_z$$

Also, from the boundary condition

$$\bar{H}_{1t} - \bar{H}_{2t} = 0$$

we can write

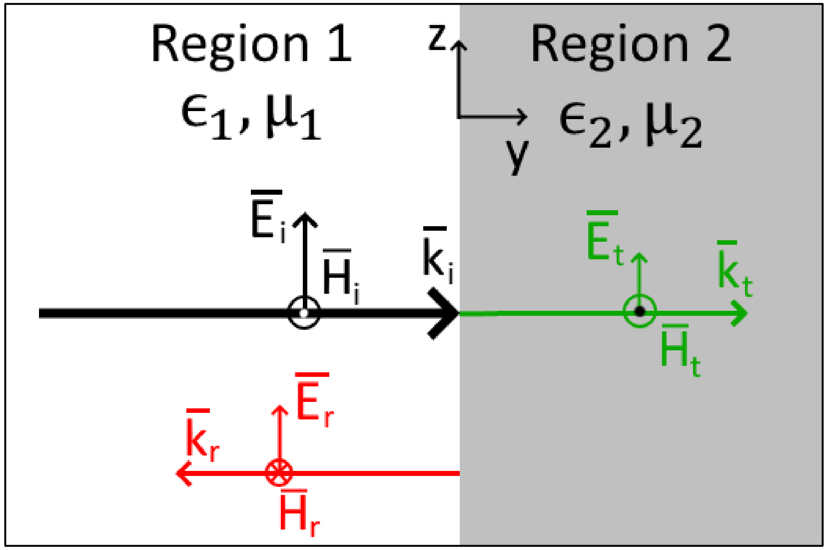
$$\bar{H}_i + \bar{H}_r = \bar{H}_t$$

or:

$$\frac{E_{oi}}{\eta_1} e^{-j\gamma_1 y} \hat{a}_z - \frac{E_{or}}{\eta_1} e^{+j\gamma_1 y} \hat{a}_z = \frac{E_{ot}}{\eta_2} e^{-j\gamma_2 y} \hat{a}_z$$

Plane Waves at a Boundary

Case 1: Normal Incidence



The two boxed equations on the previous slide may be rearranged to obtain the normal-incidence reflection and transmission coefficients:

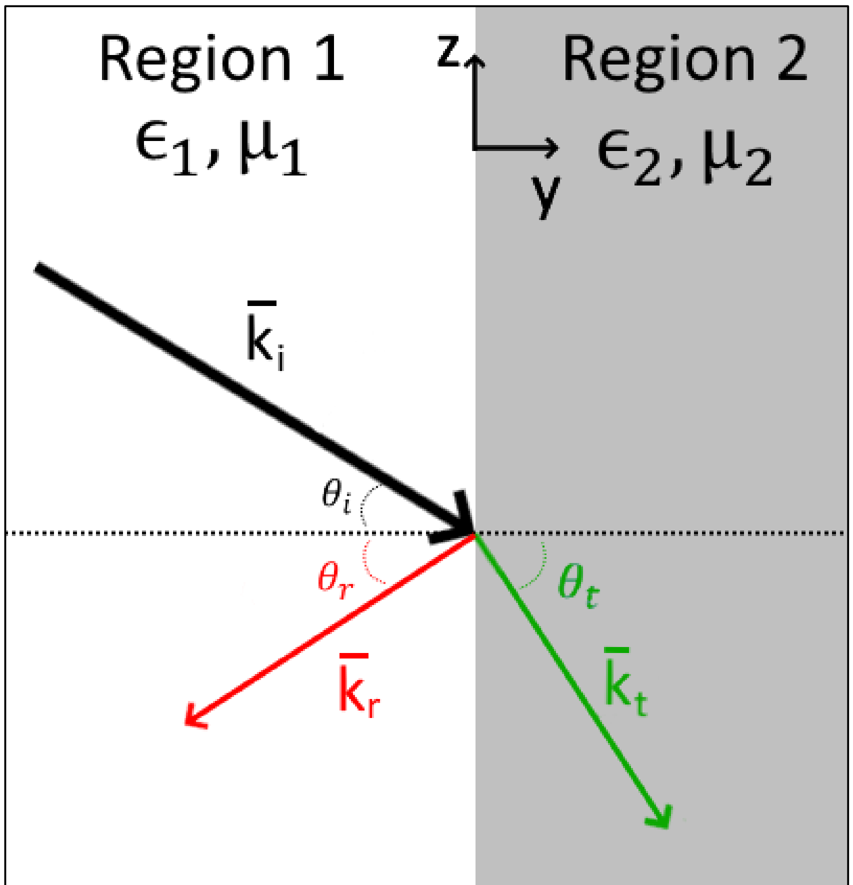
$$\Gamma_n = \frac{E_{or}}{E_{oi}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$

$$\tau_n = \frac{E_{ot}}{E_{oi}} = 1 + \Gamma_n = \frac{2\eta_2}{\eta_2 + \eta_1}$$

Plane Waves at a Boundary

Snell's Law

The next two cases will involve oblique incidence, so we will need to recall and use Snell's Law. Snell's Law gives the angles of reflected and transmitted rays at a dielectric boundary, given the angle of the incident rays and the material properties of the two dielectrics, as follows:



$$\theta_r = \theta_i$$

$$\frac{\sin(\theta_t)}{\sin(\theta_i)} = \frac{n_1}{n_2}$$

where n_1 and n_2 are the **indices of refraction** for the two dielectrics, which may be calculated from the relative permeability and permittivity values as:

$$n_1 = \sqrt{\mu_{r1}\varepsilon_{r1}}$$

$$n_2 = \sqrt{\mu_{r2}\varepsilon_{r2}}$$

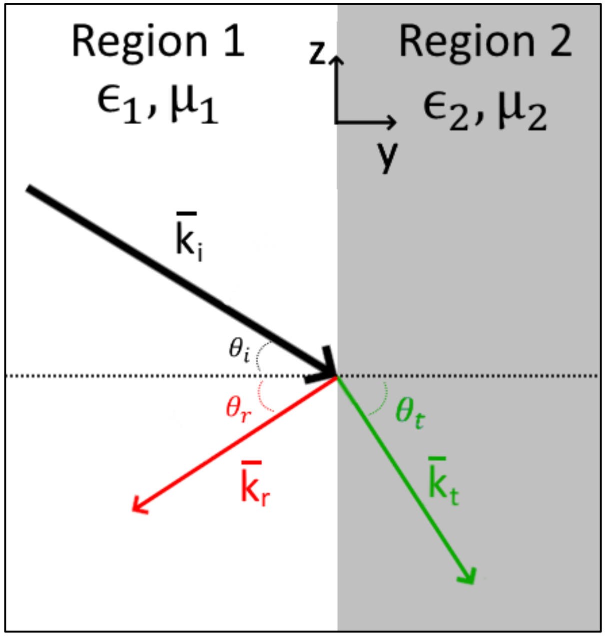
Pay attention! These are the **relative** permeability and permittivity values!

Plane Waves at a Boundary

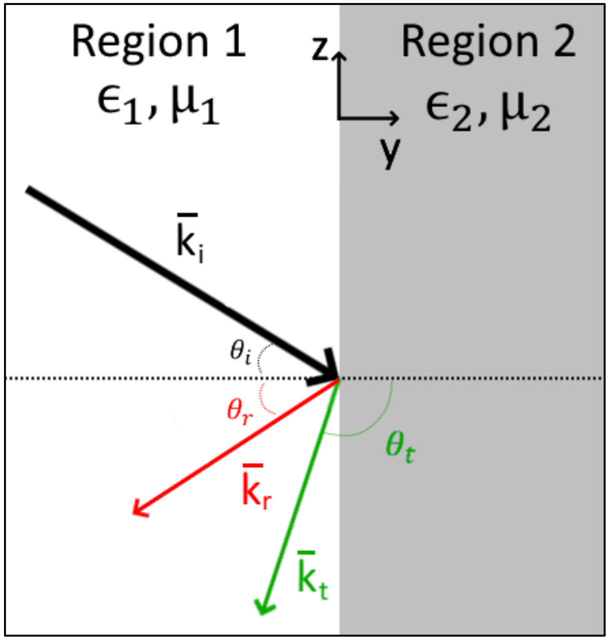
Total Internal Reflection and Critical Angle

Note: as θ_i increases, so does θ_t. If θ_t is sufficiently large (θ_t>90°), *all* the energy will end up being reflected back into region 1. This is called **total internal reflection**. It will only happen if $n_1 > n_2$ **and** $\theta_i > \theta_c$. Here, θ_c is called the "critical angle," and is calculated by:

$$\theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right)$$



General Case

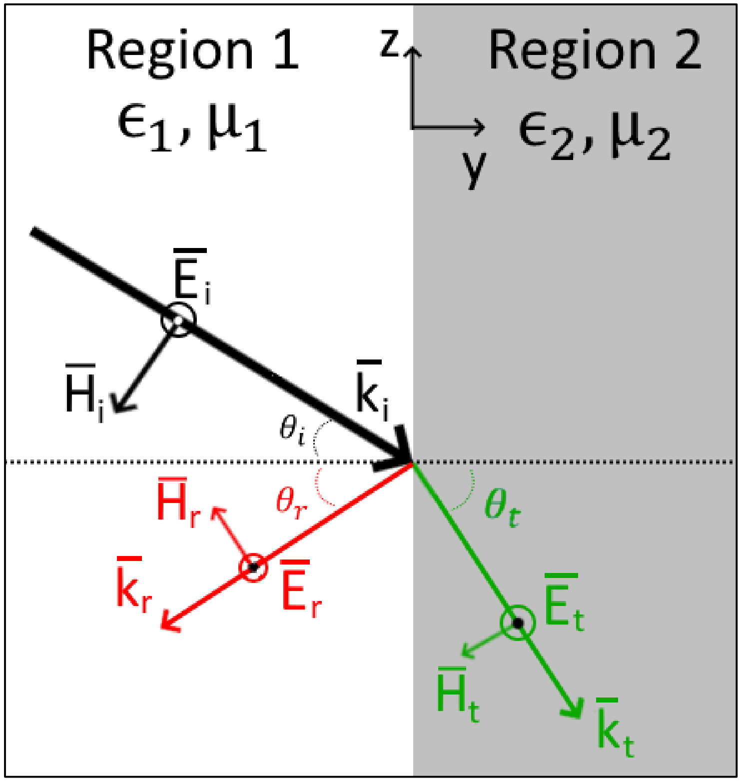


Total Internal Reflection

Plane Waves at a Boundary

Case 2: Oblique incidence, electric field tangential to boundary

Suppose a plane wave $\overline{E}_i = E_{oi}e^{-j\gamma_1[y\cos(\theta_i)-z\sin(\theta_i)]}\hat{a}_x$ is incident upon a planar boundary in the x-z plane, at an incident angle θ_i to the surface of the boundary:



In this case, the incident electric field $\overline{E}_i$ is perfectly tangential to the boundary, but the incident magnetic field $\overline{H}_i$ has both a tangential component and a normal component.

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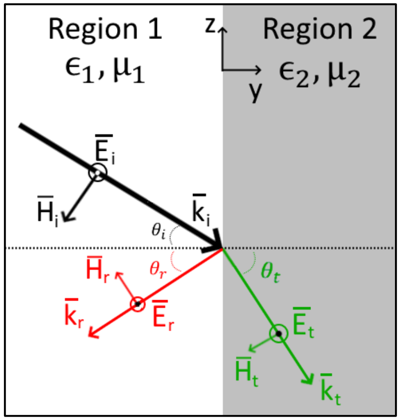
Case 2: Oblique incidence, electric field tangential to boundary



In this case, we can write:

$$\bar{E}_i + \bar{E}_r = \bar{E}_t$$

or:

$$E_{oi} e^{-\gamma_1[y\cos(\theta_i) - z\sin(\theta_i)]} \hat{a}_x$$
$$+ E_{or} e^{-\gamma_1[-y\cos(\theta_r) - z\sin(\theta_r)]} \hat{a}_x$$
$$= E_{ot} e^{-\gamma_2[y\cos(\theta_t) - z\sin(\theta_t)]} \hat{a}_x$$

And we can also write:

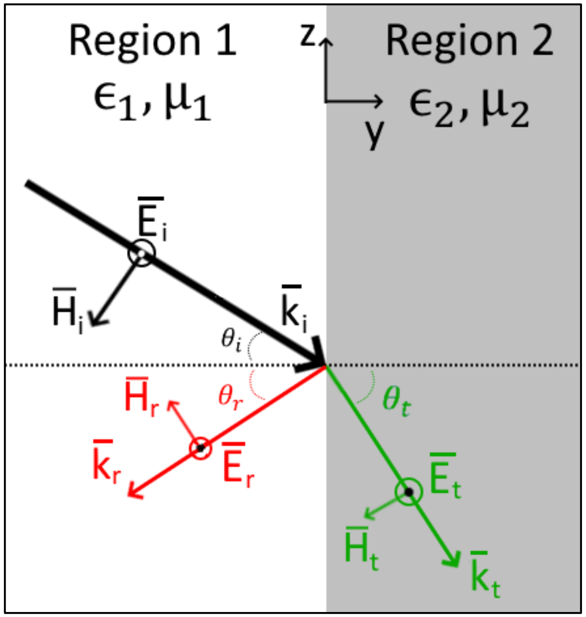
$$\bar{H}_i + \bar{H}_r = \bar{H}_t$$

or:

$$\frac{E_{oi}}{\eta_1} e^{-\gamma_1[y\cos(\theta_i) - z\sin(\theta_i)]} \left[-\sin(\theta_i)\,\hat{a}_y - \cos(\theta_i)\,\hat{a}_z\right] \cdot \hat{a}_z$$
$$+ \frac{E_{or}}{\eta_1} e^{-\gamma_1[-y\cos(\theta_r) - z\sin(\theta_r)]} [-\sin(\theta_r)\,\hat{a}_y + \cos(\theta_r)\,\hat{a}_z] \cdot \hat{a}_z$$
$$= \frac{E_{ot}}{\eta_2} e^{-\gamma_2[y\cos(\theta_t) - z\sin(\theta_t)]} [-\sin(\theta_t)\,\hat{a}_y - \cos(\theta_t)\,\hat{a}_z] \cdot \hat{a}_z$$

Plane Waves at a Boundary

Case 2: Oblique incidence, electric field tangential to boundary



The two boxed equations on the previous slide may be rearranged, with application of Snell's Law, to obtain the oblique-incidence reflection and transmission coefficients, for the case where the electric field is perfectly tangential to the boundary:

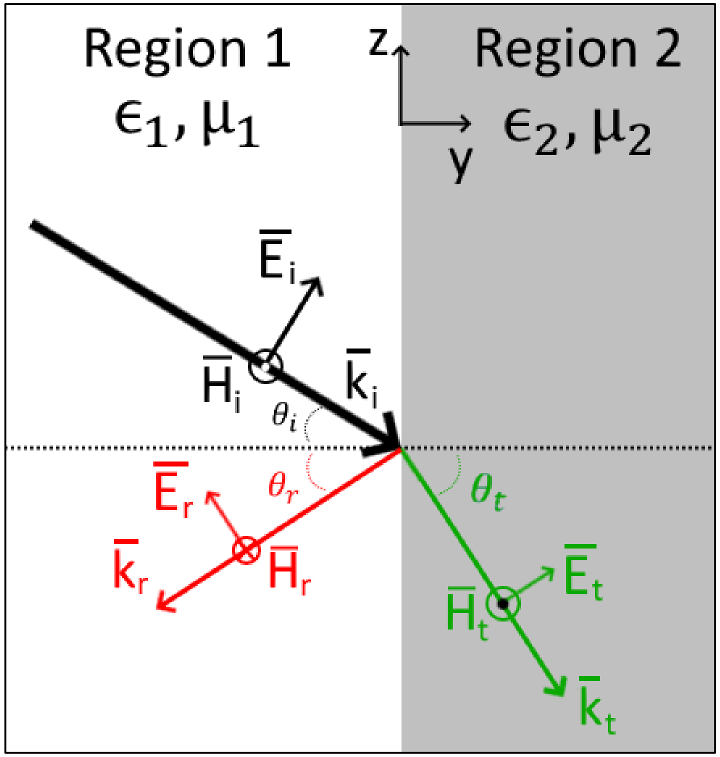
$$\Gamma_E = \frac{E_{or}}{E_{oi}} = \frac{\eta_2\cos(\theta_i) - \eta_1\cos(\theta_t)}{\eta_2\cos(\theta_i) + \eta_1\cos(\theta_t)}$$

$$\tau_E = \frac{E_{ot}}{E_{oi}} = 1 + \Gamma_n = \frac{2\eta_2\cos(\theta_i)}{\eta_2\cos(\theta_i) + \eta_1\cos(\theta_t)}$$

Plane Waves at a Boundary

Case 3: Oblique incidence, magnetic field tangential to boundary

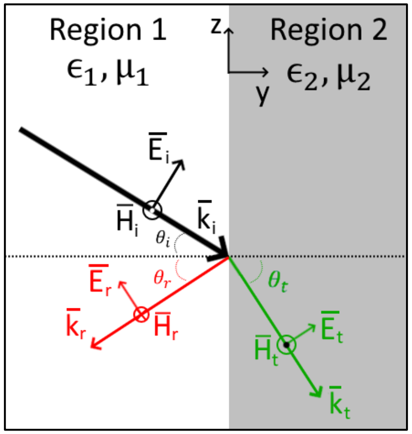
Suppose a plane wave $\overline{E}_i = E_{oi}e^{-j\gamma_1[y\cos(\theta_i)-z\sin(\theta_i)]}\left[\sin(\theta_i)\,\hat{a}_y + \cos(\theta_i)\,\hat{a}_z\right]$ is incident upon a planar boundary in the x-z plane, at an incident angle θ_i to the surface of the boundary:



In this case, the incident magnetic field $\overline{H}_i$ is perfectly tangential to the boundary, but the incident electric field $\overline{E}_i$ has both a tangential component and a normal component.

Plane Waves at a Boundary

Case 3: Oblique incidence, magnetic field tangential to boundary



In this case, we can write:

or: $\bar{E}_i + \bar{E}_r = \bar{E}_t$

$$E_{oi} e^{-\gamma_1[y\cos(\theta_i) - z\sin(\theta_i)]} \left[\sin(\theta_i)\, \hat{a}_y + \cos(\theta_i)\, \hat{a}_z\right] \cdot \hat{a}_z$$
$$+ E_{or} e^{-\gamma_1[-y\cos(\theta_r) - z\sin(\theta_r)]} \left[-\sin(\theta_r)\, \hat{a}_y + \cos(\theta_r)\, \hat{a}_z\right] \cdot \hat{a}_z$$
$$= E_{ot} e^{-\gamma_2[y\cos(\theta_t) - z\sin(\theta_t)]} \left[\sin(\theta_t)\, \hat{a}_y + \cos(\theta_t)\, \hat{a}_z\right] \cdot \hat{a}_z$$

And we can also write:

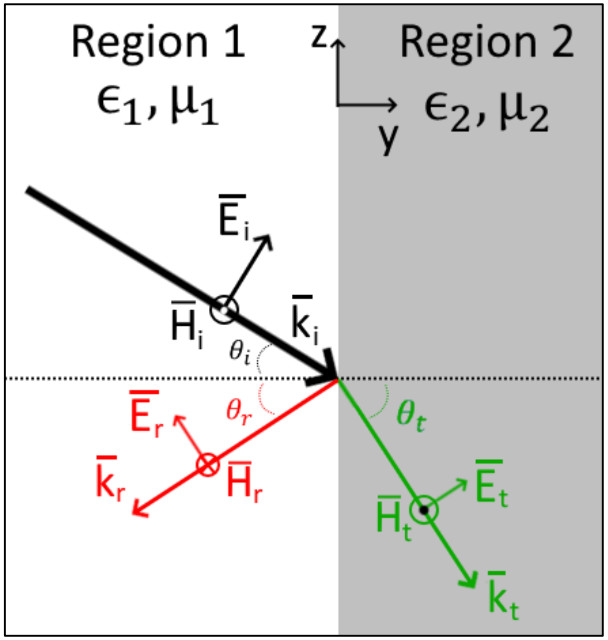
$\bar{H}_i + \bar{H}_r = \bar{H}_t$

or:

$$\frac{E_{oi}}{\eta_1} e^{-\gamma_1[y\cos(\theta_i) - z\sin(\theta_i)]} \hat{a}_x$$
$$- \frac{E_{or}}{\eta_1} e^{-\gamma_1[-y\cos(\theta_r) - z\sin(\theta_r)]} \hat{a}_x$$
$$= \frac{E_{ot}}{\eta_2} e^{-\gamma_2[y\cos(\theta_t) - z\sin(\theta_t)]} \hat{a}_x$$

Plane Waves at a Boundary

Case 3: Oblique incidence, magnetic field tangential to boundary



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$$\Gamma_H = \frac{E_{or}}{E_{oi}} = \frac{\eta_1 \cos(\theta_i) - \eta_2 \cos(\theta_t)}{\eta_1 \cos(\theta_i) + \eta_2 \cos(\theta_t)}$$

$$\tau_H = \frac{E_{ot}}{E_{oi}} = \frac{2\eta_2 \cos(\theta_i)}{\eta_2 \cos(\theta_t) + \eta_1 \cos(\theta_i)}$$

The Poynting Theorem

*Electromagnetic waves carry **energy.** The Poynting Theorem gives us insight into energy storage and transfer in the context of electromagnetics*

Derivation of the Poynting Theorem:

$$\nabla \times \overline{\mathcal{E}} = -\frac{d\overline{\mathcal{B}}}{dt} \qquad \textit{(Faraday's Law)}$$

$$\nabla \times \overline{\mathcal{H}} = \frac{d\overline{\mathcal{D}}}{dt} + \overline{\mathcal{J}} \qquad \textit{(Ampere's Law)}$$

so that:

$$\overline{H} \cdot [\nabla \times \overline{\mathcal{E}}] - \overline{\mathcal{E}} \cdot [\nabla \times \overline{\mathcal{H}}] = \overline{\mathcal{H}} \cdot \left[-\frac{d\overline{\mathcal{B}}}{dt}\right] - \overline{\mathcal{E}} \cdot \left[\frac{d\overline{\mathcal{D}}}{dt} + \overline{\mathcal{J}}\right]$$

which can be rearranged, using vector identities, as:

$$\nabla \cdot [\overline{\mathcal{E}} \times \overline{\mathcal{H}}] = -\overline{\mathcal{H}} \cdot \frac{d\overline{\mathcal{B}}}{dt} - \overline{\mathcal{E}} \cdot \frac{d\overline{\mathcal{D}}}{dt} - \overline{\mathcal{E}} \cdot \overline{\mathcal{J}}$$

$Note: we\ are\ using\ the\ script\ symbols$ $\overline{\mathcal{E}}, \overline{\mathcal{H}}, \overline{\mathcal{B}}, \overline{\mathcal{D}}$ and $\overline{\mathcal{J}}$ to specify the time-domain expressions of $\overline{E}$, $\overline{H}$, $\overline{B}$, $\overline{D}$, and $\overline{J}$, respectively.

The Poynting Theorem

Electromagnetic waves carry ***energy.*** *The Poynting Theorem gives us insight into energy storage and transfer in the context of electromagnetics*

Derivation of the Poynting Theorem:

$$\nabla \cdot \left[\overline{\mathcal{E}} \times \overline{\mathcal{H}}\right] = -\overline{\mathcal{H}} \cdot \frac{d\overline{\mathcal{B}}}{dt} - \overline{\mathcal{E}} \cdot \frac{d\overline{\mathcal{D}}}{dt} - \overline{\mathcal{E}} \cdot \overline{\mathcal{J}}$$

If we consider a region of volume *V,* contained by surface *S,* this can be rewritten, using the divergence theorem, as:

$$\oint_S \left(\overline{\mathcal{E}} \times \overline{\mathcal{H}}\right) \cdot \overline{dS} = -\int_V \frac{d}{dt}\left(\frac{\mu}{2}\left|\overline{\mathcal{H}}\right|^2 + \frac{\varepsilon}{2}\left|\overline{\mathcal{E}}\right|^2\right) - \int_V \overline{\mathcal{E}} \cdot \overline{\mathcal{J}}$$

This is the Poynting Theorem

The Poynting Theorem

But, what does it mean? Let's look at a few of these pieces individually…

$\frac{\mu}{2}\left|\overline{\mathcal{H}}\right|^2$ **This is the energy stored in the magnetic fields in the volume *V***

$\frac{\varepsilon}{2}\left|\overline{\mathcal{E}}\right|^2$ **This is the energy stored in the electric fields in the volume *V***

$\oint_S \left(\overline{\mathcal{E}} \times \overline{\mathcal{H}}\right) \cdot \overline{dS}$ **This is the power per area leaving the volume through surface S**

$\int_V \overline{\mathcal{E}} \cdot \overline{\mathcal{J}}$ **This is the power consumed by ohmic losses inside the volume *V***

So, the Poynting Theorem says that net power flow into a volume must either be stored in the internal fields or consumed by ohmic losses.

Alternatively, the Poynting Theorem says that net power flow out of a volume must be being released from internal fields

The Poynting Theorem is a statement of Conservation of Energy

The Poynting Vector

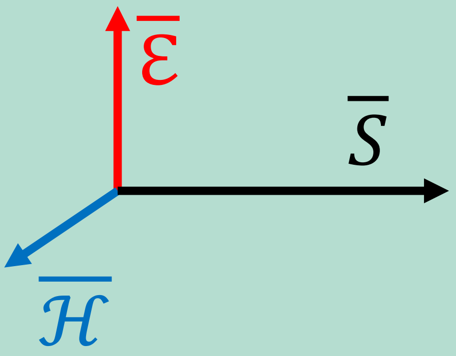
We will define the Poynting Vector as:

$$\overline{S} = \overline{\mathcal{E}} \times \overline{\mathcal{H}}$$

The vector $\overline{S}$ points in the direction of power flow, and has units of W/m^2, indicating the power flow density of the electromagnetic wave.

This is an expression of *instantaneous* power flow (both $\overline{\mathcal{E}}$ and $\overline{\mathcal{H}}$ depend on time)

Notice the implied right-handed triad:



The Poynting Vector

It is sometimes more useful to consider the time-average power flow density. The time-average Poynting vector is given by:

$$\overline{S_{av}} = \frac{1}{2}\mathcal{R}e\left\{\overline{E} \times \overline{H}^*\right\}$$

where $\overline{E}$ and $\overline{H}$ are now in phasor form, and $\overline{H}^*$ indicates the complex conjugate of $\overline{H}$.

The vector $\overline{S_{av}}$ still points in the direction of power flow, and has units of W/m^2, indicating the power flow density of the electromagnetic wave.

This is an expression of the *time-averaged* power flow density.

This is still a right-handed triad: